Multi-qubit stabilizer and cluster entanglement witnesses

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Abstract

One of the problems concerning entanglement witnesses (EWs) is the construction of them by a given set of operators. Here several multi-qubit EWs called stabilizer EWs are constructed by using the stabilizer operators of some given multi-qubit states such as GHZ, cluster and exceptional states. The general approach to manipulate the multi-qubit stabilizer EWs by exact(approximate) linear programming (LP) method is described and it is shown that the Clifford group play a crucial role in finding the hyper-planes encircling the feasible region. The optimality, decomposability and non-decomposability of constructed stabilizer EWs are discussed.

Keywords: Entanglement Witness, Stabilizer group, Clifford Group, Linear Programming, Feasible Region.

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1 Introduction

The problem of characterizing n-qubit entanglement has motivated considerable interest in the literature. This problem was raised within the context of quantum information and quantum computation processes such as teleportation, dense coding and quantum key distribution [1, 2, 3] which consider the physical phenomenon of entanglement as a resource. Though there are a number of very useful and spectacular results for detecting the presence of entanglement in pure and mixed states of multipartite systems, the subject is still at its infancy [4, 5, 6, 7].

Among the different criteria to analyze the separability of quantum states the entanglement witnesses (EWs) are of special interest since it has been proved that for any entangled state there exists at least one EW detecting it [8, 9]. The EWs are Hermitian operators which have non-negative expectation values over all separable states and detect some entangled states. A great deal of investigation has been devoted to the study of EWs, considering their decomposability, optimality [11], optimal setups for local measurements of witnesses [12, 13] and even their use in the characterization of entanglement in important physical systems [14, 15, 16]. Inside the several problems concerning the EWs, the problem of how to construct EWs by a given set of operators has a great importance. From a different point of view, a very useful approach to construct EWs is the linear programming (LP) [17, 18, 19, 20, 21, 22], a special case of convex optimization which can be solved by using very efficient algorithms such as the simplex and interior-point methods (see e.g. [23, 24]). In fact, in order to a hermitian operator W be an EW, it must posses at least one negative eigenvalue and the expectation value of W over any separable state must be non-negative. Therefore, for determination of EWs, one needs to determine the minimum value of this expectation value over the feasible region (the minimum value must be non-negative) and hence the problem reduces to an optimization over the convex set of feasible region. For example, in [19, 20] the manipulation of generic Bell-states diagonal EWs has been reduced to such an optimization problem. It has been

shown that, if the feasible region for this optimization problem constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly the optimization problem. This problem is called linear programming (LP) and the simplex method is the easiest way of solving it. If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically, one can define a new convex hull which is a polygon and has encircled the feasible region. The points on the boundary of the polygon can approximately determine the minimum value of the optimization problem. Thus the approximated value is obtained via LP. In general, it is difficult to find this region and solve the corresponding optimization problem; thus, it is difficult to find any generic multipartite EW. Recently, in Ref. [21], a new class of EWs called reduction type EWs has been introduced for which the feasible regions turn out to be convex polygons. Also, in Ref.[22], some kinds of Bell-states diagonal relativistic multispinor EWs have been constructed which can be manipulated by using exact and approximate LP method.

On the other hand, stabilizer formalism and Clifford group operations have been proved to be useful in quantum error correction (theory of stabilizer codes) [25, 26, 27, 28], quantum computing, entanglement distillation [29, 30, 31] and entanglement detection [12, 13, 14]. In this paper, we link stabilizer theory and Clifford group operations with structure of new type EWs, the so-called stabilizer EWs (SEWs). As we will show all vertex points and hyper-planes surrounding feasible regions (i.e., the regions coming from the positivity of EWs with separable states) can be obtained just from a few ones by applying the Clifford group operations. The optimality of SEWs corresponding to hyper-planes surrounding feasible region is discussed in detail and it is shown that the optimality has a close connection with the common eigenvectors of stabilizer operators.

The paper is organized as follows: In Section 2, we review the basic notions and definitions of EWs relevant to our study and describe a general approach of constructing stabilizer EWs by exact and approximate LP method. In Section 3, we consider the construction of SEWs

that can be solved by exact LP method and as instances of such SEWs, we describe the SEWs of GHZ and cluster states in details and give a brief discussion about SEWs of five, seven, eight and nine qubit stabilizer states. Also the role of Clifford group operations is studied in this construction. Section 4 is devoted to an analysis of optimality of the introduced SEWs. It is proved that some of the SEWs which correspond to surrounding half-planes of the feasible regions are optimal. In Section 5, we consider the decomposability or non-decomposability of GHZ and cluster states SEWs and show that the three-qubit SEWs are all decomposable but for more than three-qubit, there exist non-decomposable SEWs as well. In Section 6, we give some entangled mixed states that can be detected by the SEWs. Section 7 is devoted to construct SEWs that their feasible regions are not polygons by themselves but can be approximated by polygons and then solved by LP method. The paper is ended with a brief conclusion and two appendices.

2 Stabilizer EWs and LP method

2.1 Entanglement witnesses

First let us recall the definition of entanglement and separability. An n-partite quantum mixed state $\rho \in \mathcal{B}(\mathcal{H})$ (the Hilbert space of bounded operators acting on $\mathcal{H} = \mathcal{H}_{d_1} \otimes ... \otimes \mathcal{H}_{d_n}$) is called fully separable if it can be written as a convex combination of pure product states, that is

$$\rho = \sum_i p_i |\alpha_i^{(1)}\rangle\langle\alpha_i^{(1)}| \otimes |\alpha_i^{(2)}\rangle\langle\alpha_i^{(2)}| \otimes ... \otimes |\alpha_i^{(n)}\rangle\langle\alpha_i^{(n)}| \tag{2.1}$$

where $|\alpha_i^{(j)}\rangle$ with $j = 1, ..., n$ are arbitrary but normalized vectors lying in the $\mathcal{H}_{d_j}$, and $p_i \geq 0$ with $\sum_i p_i = 1$. When this is not the case, ρ is called entangled. Although the definitions of separable and entangled states were extended to consider various partitions of the original system into subsystems [33, 34], throughout the paper by separability we mean

fully separability.

An entanglement witness $\mathcal{W}$ is a Hermitian operator such that $Tr(\mathcal{W}\rho_s) \geq 0$ for all separable states ρ_s and there exists at least one entangled state ρ_e which can be detected by $\mathcal{W}$, that is $Tr(\mathcal{W}\rho_e) < 0$. Note that in the aforementioned definition of EWs, we are not worry about the kind of entanglement of the quantum state and we are rather looking for EWs which have non-negative expectation values over all separable states despite the fact that they have some negative eigenvalues. The existence of an EW for any entangled state is a direct consequence of Hahn-Banach theorem [32] and the fact that the set of separable density operators is convex and closed.

Based on the notion of partial transposition, the EWs are classified into two classes: decomposable (d-EW) and non-decomposable (nd-EW). An EW $\mathcal{W}$ is called decomposable if there exist positive operators $\mathcal{P}$, $\mathcal{Q}_K$ such that

$$\mathcal{W} = \mathcal{P} + \sum_{K \subset \mathcal{N}} \mathcal{Q}_K^{T_K} \tag{2.2}$$

where $\mathcal{N} := \{1, 2, 3, ..., n\}$ and T_K denotes the partial transposition with respect to partite $K \subset \mathcal{N}$ and it is non-decomposable if it can not be written in this form [17]. Clearly d-EW can not detect bound entangled states (entangled states with positive partial transpose (PPT) with respect to all subsystems) whereas there are some bound entangled states which can be detected by an nd-EW.

Usually one is interested in finding EWs $\mathcal{W}$ which detect entangled states in an optimal way in the sense that when we subtract any positive operator from $\mathcal{W}$, then it does not remain an EW anymore [5]. In other words, if there exist $\epsilon > 0$ and a positive operator $\mathcal{P}$ such that $\mathcal{W}' = \mathcal{W} - \epsilon\mathcal{P}$ is again an EW, then we conclude that $\mathcal{W}$ is not optimal and otherwise it is an optimal EW.

2.2 Manipulation of EWs by exact and approximate LP method

This subsection is devoted to describe linear programming (LP) and general approach to manipulate the so-called stabilizer EWs by exact or approximate LP method [23].

Consider a non-positive Hermitian operator of the form

$$\mathcal{W} = a_0 I + \sum_i a_i Q_i \tag{2.3}$$

where Q_i are Hermitian operators and a_i's are real parameters with $a_0 > 0$. In this work, the operators Q_i will be considered as operations of a given multi-qubit stabilizer group. The stabilizer operations are mutually commuting and their eigenvalues are +1 and -1. We will attempt to choose the real parameters a_i such that $\mathcal{W}$ becomes an EW. To this aim, we introduce the maps

$$P_i = Tr(Q_i \rho_s) \tag{2.4}$$

for any separable state ρ_s. The maps P_i map the convex set of separable states into a region which will be named feasible region. Since $-1 \leq P_i \leq 1$ for all i, the feasible region is bounded and lies inside the hypercube defined by $-1 \leq P_i \leq 1$ for all i. The first property of an EW is that its expectation value over any separable state is non-negative, i.e., the condition

$$\mathcal{F}_{\mathcal{W}} := Tr(\mathcal{W}\rho_s) = a_0 + \sum_i a_i P_i \geq 0$$

is satisfied for any point of the feasible region. For satisfying this condition, it is sufficient that the minimum value of $\mathcal{F}_{\mathcal{W}}$ be non-negative. Therefore, for determination of EWs of type (2.3), one needs to determine the minimum value of $a_0 + \sum_{i=1}^{n} a_i P_i$ over the feasible region (the minimum value must be non-negative) and hence the problem reduces to the optimization of the linear function $a_0 + \sum_{i=1}^{n} a_i P_i$ over the convex set of feasible region.

We note that, the quantity $\mathcal{F}_{\mathcal{W}}$ achieves its minimum value for pure product states, since every separable mixed state ρ_s can be written as a convex combination of pure product states,

say $\rho_s = \sum_i p_i |\Upsilon_i\rangle\langle\Upsilon_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$, whence we have

$$Tr(\mathcal{W}\rho_s) = \sum_i p_i Tr(\mathcal{W}|\Upsilon_i\rangle\langle\Upsilon_i|) \geq C_{min}, \tag{2.5}$$

$$\text{with} \qquad C_{min} = \min_{|\Upsilon\rangle \in D_{prod.}} Tr(\mathcal{W}|\Upsilon\rangle\langle\Upsilon|)$$

where, $D_{prod.}$ denotes the set of pure product states. In this work, we are interested in the EWs that their feasible regions are of simplex (or at most convex polygon) types. The manipulation of these EWs amounts to

$$\begin{array}{rl} \text{minimize} & \mathcal{F}_{\mathcal{W}} = a_0 + \sum_i a_i P_i \\ \text{subject to} & \sum_i (c_{ij} P_i - d_j) \geq 0 \quad j = 1, 2, ... \end{array} \tag{2.6}$$

where c_{ij} and d_j are parameters of hyper-planes surrounding the feasible regions. So the problem reduces to a LP problem. On the basis of LP method, minimum of an objective function $\mathcal{F}_{\mathcal{W}}$ always occurs at the vertices of bounded feasible region. Therefore the vertices of feasible region come from pure product states.

It is necessary to distinguish between two cases: (**a**) exactly soluble, and (**b**) approximately soluble EWs. In the case **a**, the boundaries (constraints on P_i) come from finite vertices arising from pure product states and construct a convex polygon, while in the case **b** the feasible region is not a polygon and the boundaries may be bounded convex hypersurfaces. In this case, with the help of tangent planes in this region at points which are determined either analytically or numerically, one can define a new convex hull which is a polygon encircling the feasible region, i.e., we approximate the boundaries with hyper-planes and clearly some vertices do not arise from pure product states. The points on the boundary of the polygon can approximately determine the minimum value of $\mathcal{F}_{\mathcal{W}}$ in (2.6). Thus the approximated value is obtained via LP. The both cases can be solved by the well-known simplex method. The simplex algorithm is a common algorithm used to solve an optimization problem with a polytope feasible region, such as a linear programming problem. It is an improvement over the algorithm to test all feasible solution of the convex feasible region and then choose the optimal feasible solution. It does this

by moving from one vertex to an adjacent vertex, such that the objective function is improved. This algorithm still guarantees that the optimal point will be discovered. In addition, only in the worst case scenario will all vertices be tested. Here, considering the scope of this paper, a complete treatment of the simplex algorithm is unnecessary; for a more complete treatment please refer to any LP text such as [23, 24].

3 Exactly soluble stabilizer EWs

In this section we consider the construction of stabilizer EWs (SEWs) which can be solved exactly by the LP method. In motivating this construction, we begin with EWs which can be constructed by the stabilizer operations of the multi-qubit GHZ state.

But before proceeding, it should be noticed that the Hermitian operator of the form (2.3) can not be a SEW when all the Q_i's form pairwise locally commuting set. Two operators

$$Q = L_1 \otimes ... \otimes L_n \quad \text{and} \quad Q' = K_1 \otimes ... \otimes K_n.$$

are called locally commuting if $[L_i, K_i] = 0$, for all $i = 1, 2, ..., n$. To prove this assertion, consider the following operator

$$\mathcal{W} = a\ I + b\ Q + c\ Q'.$$

Because of the commutativity of K_i and L_i we have the

$$L_i = \sum_{\nu_i} \lambda^{(i)}_{\nu_i} |\psi^{(i)}_{\nu_i}\rangle\langle\psi^{(i)}_{\nu_i}| \qquad , \qquad K_i = \sum_{\nu_i} \mu^{(i)}_{\nu_i} |\psi^{(i)}_{\nu_i}\rangle\langle\psi^{(i)}_{\nu_i}|.$$

which in turn imply that the operator $\mathcal{W}$ can be written as

$$\mathcal{W} = a\ I + b\ \bigotimes_{i=1}^{n} \sum_{\nu_i} \lambda^{(i)}_{\nu_i} |\psi^{(i)}_{\nu_i}\rangle\langle\psi^{(i)}_{\nu_i}| + c\ \bigotimes_{i=1}^{n} \sum_{\nu_i} \mu^{(i)}_{\nu_i} |\psi^{(i)}_{\nu_i}\rangle\langle\psi^{(i)}_{\nu_i}|$$

$$= \sum_{\nu_1} ... \sum_{\nu_n} (a + b\lambda^{(1)}_{\nu_1} ... \lambda^{(n)}_{\nu_n} + c\mu^{(1)}_{\nu_1} ... \mu^{(n)}_{\nu_n})\ \bigotimes_{i=1}^{n} |\psi^{(i)}_{\nu_i}\rangle\langle\psi^{(i)}_{\nu_i}|$$

Now if we want $\mathcal{W}$ to be an EW then it must has non-negative expectation values with all pure product states which means that all eigenvalues $(a + b\lambda^{(1)}_{\nu_1}...\lambda^{(n)}_{\nu_n} + c\mu^{(1)}_{\nu_1}...\mu^{(n)}_{\nu_n})$ are non-negative, hence $\mathcal{W}$ is a positive operator. Therefore, the SEWs can be constructed from the set of stabilizer operators Q_i that at least one pair of them is not locally commuting.

Throughout the paper, the generators of stabilizer groups are chosen according to the table of appendix I. Of course, this choice is arbitrary and one can take other elements as generators. By the method presented here we can construct SEWs (exactly or approximately) for completely different stabilizer groups.

3.1 GHZ stabilizer EWs

We consider even case of GHZ SEWs which lies in realm of exactly soluble LP problems. The odd case is discussed in appendix III. A similar construction can be made based on other elements of the GHZ stabilizer group.

3.1.1 Even case

Let us consider a situation in which the Hermitian operator is composed of all generators of GHZ stabilizer group together with all even terms $S^{(\mathrm{GHZ})}_1 S^{(\mathrm{GHZ})}_{2k}$ (the name even refer to the index $2k$) as follows

$$\mathcal{W}^{(n)}_{GHZ} = a_0 I_{2^n} + \sum_{k=1}^{n} a_k S^{(\mathrm{GHZ})}_k + \sum_{k=1}^{n'} a_{1,2k} S^{(\mathrm{GHZ})}_1 S^{(\mathrm{GHZ})}_{2k} \quad , \quad n' := \left[\frac{n}{2}\right], \tag{3.7}$$

where, $S^{(\mathrm{GHZ})}_k$ for $k = 1, ..., n$ are given in the table of the Appendix I and the reader is referred to that appendix for an overview of the stabilizer formalism. Due to the commutativity of all GHZ stabilizer generators, it is easy to see that the eigenvalues of $\mathcal{W}^{(n)}_{GHZ}$ are

$$a_0 + \sum_{k=1}^{n} (-1)^{i_k} a_k + \sum_{k=1}^{n'} (-1)^{i_1 + i_{2k}} a_{1,2k} \quad , \quad \forall\ (i_1, i_2, ..., i_n) \in \{0, 1\}^n. \tag{3.8}$$

Evidently, when all eigenvalues are positive the above operator is positive; otherwise it may be a SEW.

For a separable state ρ_s, the positivity of

$$Tr(\mathcal{W}^{(n)}_{_{GHZ}}\rho_s) \geq 0$$

implies the positivity of the objective function

$$\mathcal{F}_{\mathcal{W}^{(n)}_{GHZ}} = a_{_0} + \sum_{k=1}^{n} a_{_k} P_{_k} + \sum_{k=1}^{n'} a_{_{1,2k}} P_{_{1,2k}} \geq 0, \tag{3.9}$$

where

$$P_{_k} = Tr(S^{(\mathrm{GHZ})}_{_k}\rho_s) \quad , \quad P_{_{1,2k}} = Tr(S^{(\mathrm{GHZ})}_{_1} S^{(\mathrm{GHZ})}_{_{2k}}\rho_s),$$

and all of the $P_{_k}$'s and $P_{_{1,2k}}$'s lie in the interval $[-1,1]$. Furthermore, the operator $\mathcal{W}^{(n)}_{_{GHZ}}$ must has at least one negative eigenvalue to become a SEW. To reduce the problem to a LP one and to determine the feasible region, we require to know the vertices, namely the extreme points of the feasible region. Vertex points of the feasible region come from pure product states. The coordinates of vertex points can take one of three values +1, -1 and 0. Regarding the above considerations, the product vectors and the vertex points of the feasible region coming from them are listed in table 1, where

Product state	$(P_2, P_3, ..., P_{n-1}, P_n, P_1, P_{1,2}, P_{1,4}, ..., P_{1,2n'-2}, P_{1,2n'})$
$\vert\Psi^{\pm}\rangle$	$(0,0,...,0,0,\pm1,0,0,...,0,0)$
$\Lambda_{_1}\vert\Psi^{\pm}\rangle$	$(0,0,...,0,0,0,\pm1,0,...,0,0)$
$\vdots$	$\vdots$
$\Lambda_{_{n'}}\vert\Psi^{\pm}\rangle$	$(0,0,...,0,0,0,0,0,...,0,\pm1)$
$\Xi_{i_2,...,i_n}\vert\Psi^{+}\rangle$	$\big((-1)^{i_2},(-1)^{i_2+i_3},...,(-1)^{i_{n-2}+i_{n-1}},(-1)^{i_{n-1}+i_n},0,0,0,...,0,0\big)$

Table 1: The product vectors and coordinates of vertices for $\mathcal{W}^{(n)}_{_{GHZ}}$.

$$\begin{gathered}
|\Psi^{\pm}\rangle = |x^{\pm}\rangle_{_1}|x^{+}\rangle_{_2}|x^{+}\rangle_{3}...|x^{+}\rangle_{_n} \\
\Lambda_{_k} = \left(M^{(2k-1)}\right)^{\dagger} M^{(2k)} \quad k=1,2,...,n' \\
\Xi_{i_2,...,i_n} = (\sigma^{(2)}_{_x})^{i_2}...(\sigma^{(n)}_{_x})^{i_n} \bigotimes_{j=1}^{n} H^{(j)} \quad , \quad \forall\ (i_2,i_3,...,i_n) \in \{0,1\}^{n-1}
\end{gathered} \tag{3.10}$$

and $|x^{\pm}\rangle$ are eigenvectors of σ_x with eigenvalues ± 1. Here $M^{(k)}$ and $H^{(k)}$ are the phase-shift operator and Hadamard transform acting on particle k respectively (see appendix I). One can easily check by direct calculation that the convex hull of the points listed in table 1 is contained in the feasible region and form a $(n-1)2^{n'+2}$-simplex with the following boundary hyper-planes

$$|P_{1} \pm P_{j} + \sum_{k=1}^{n'} (-1)^{i_k} P_{1,2k}| = 1, \quad j = 2, ..., n, \quad \forall\ (i_1, i_2, ..., i_{n'}) \in \{0,1\}^{n'}.$$

On the other hand, in appendix II it is shown that the feasible region is also contained in this simplex, i.e., the feasible region is exactly determined by the intersection of the half-spaces

$$|P_{1} \pm P_{j} + \sum_{k=1}^{n'} (-1)^{i_k} P_{1,2k}| \leq 1. \tag{3.11}$$

In fact the half-spaces (3.11) come from the positivity of the expectation values of the operators

$$\begin{array}{l} I_{2^n} + S_{1}^{(\mathrm{GHZ})} \pm S_{j}^{(\mathrm{GHZ})} + \sum_{k=1}^{n'} (-1)^{i_k} S_{1,2k}^{(\mathrm{GHZ})} \\ I_{2^n} - S_{1}^{(\mathrm{GHZ})} \pm S_{j}^{(\mathrm{GHZ})} - \sum_{k=1}^{n'} (-1)^{i_k} S_{1,2k}^{(\mathrm{GHZ})} \end{array} \quad , \quad j = 2, ..., n \quad , \quad \forall\ (i_1, i_2, ..., i_{n'}) \in \{0,1\}^{n'}$$

over pure product states. We note that it is not necessary to consider all the above operators since one can obtain them just by applying some elements of the Clifford group (see Appendix I) on the $2n' + n''$ (compare with $(n-1)2^{n'+2}$) following operators

$$\begin{array}{ll} I_{2^n} \pm \left(S_{1}^{(\mathrm{GHZ})} + S_{2j}^{(\mathrm{GHZ})} + \sum_{k=1}^{n'} S_{1,2k}^{(\mathrm{GHZ})}\right) & j = 1, ..., n' \\ I_{2^n} - S_{1}^{(\mathrm{GHZ})} - S_{2j+1}^{(\mathrm{GHZ})} - \sum_{k=1}^{n'} S_{1,2k}^{(\mathrm{GHZ})} & j = 1, ..., n''. \end{array} \tag{3.12}$$

For example we get the operator $S = I_{2^n} + S_{1}^{(\mathrm{GHZ})} - S_{2j}^{(\mathrm{GHZ})} - S_{1,2j}^{(\mathrm{GHZ})} + \sum_{k \neq j}^{n'} S_{1,2k}^{(\mathrm{GHZ})}$ from the operator $S' = I_{2^n} + S_{1}^{(\mathrm{GHZ})} + S_{2j}^{(\mathrm{GHZ})} + \sum_{k=1}^{n'} S_{1,2k}^{(\mathrm{GHZ})}$ under conjugation with the Clifford operation $\sigma_x^{(2j)}$, i.e.,

$$S = \left(\sigma_x^{(2j)}\right) S' \left(\sigma_x^{(2j)}\right)^{\dagger}.$$

Now the problem of finding a pre-SEW (a hermitian operator with non-negative expectation value over any separable state) of the form (3.7) is reduced to the LP problem

$$\begin{array}{rl} \text{minimize} & \mathcal{F}_{\mathcal{W}_{GHZ}^{(n)}} = a_0 + \sum_{k=1}^{n} a_{k} P_{k} + \sum_{k=1}^{n'} a_{1,2k} P_{1,2k} \\ \text{subject to} & |P_{1} \pm P_{j} + \sum_{k=1}^{n'} (-1)^{i_k} P_{1,2k}| \leq 1, \quad j = 2, ..., n, \quad \forall\ (i_1, i_2, ..., i_{n'}) \in \{0,1\}^{n'} \end{array} \tag{3.13}$$

On the basis of LP method, minimum of an objective function always occurs at the vertices of the bounded feasible region. Hence, if we put the coordinates of the vertices (see table 1) in the objective function (3.9) and require the non-negativity of the objective function on all vertices, we get the conditions

$$\begin{array}{c} a_0 > 0 \quad , \quad a_0 \geq |a_1| \quad , \quad a_0 \geq \sum_{i=2}^{n} |a_i| \\ a_0 \geq |a_{1,2k}| \qquad k = 1, ..., n' \end{array} \tag{3.14}$$

for parameters a_i. Evidently, these conditions are sufficient to ensure that the objective function is non-negative on the whole of the feasible region. If we take $a_0 = (n-1)$, $a_k = -1$, for all $k = 1, ..., n$ and $a_{1,2k} = 0$, for all $k = 1, ..., n'$, which fulfill all the conditions of Eq. (3.14), then we get the SEW stated in Eq. (21) of Ref. [10]. Also by taking $a_0 = 1$, $a_1 = -1$ and $a_m = a_{1,m} = -1$ ($m \geq 2$ is even) we have the SEWs stated in Eq. (21) of the mentioned reference as special cases.

Fixing a_0 in the space of parameters, all of the a_i's lie inside the polygon defined by inequalities (3.14). Now in order that the operator of Eq.(3.7) becomes non-negative, all of its eigenvalues in (3.8) must be non-negative. The intersection of half-spaces arising from the non-negativity of the eigenvalues form a polyhedron inside the aforementioned polygon. The complement of this polyhedron in the polygon is the where that the operator (3.7) is SEW and will be named the SEWs region.

We assert that the SEWs region is non-empty. To confirm this assertion, we discuss the case that all parameters a_i are positive since the discussion for other cases can be easily come from by replacing any parameter by its negative value (except a_0 which is always positive). Because of the symmetry between the parameters a_1 and a_{2k} 's ($k = 1, ..., n'$), we can assume without loss of generality that $a_2 \geq a_4 \geq a_6 \geq ... \geq a_{2n'}$. With this assumption, all of the $2^{n''} + n'$ eigenvalues (with $n'' = [\frac{n-1}{2}]$):

$$\begin{cases} a_0 + a_1 + \sum_{j=1}^{n'} a_{2j} + \sum_{j=1}^{n''} (-1)^{i_{2j+1}} a_{2j+1} + \sum_{k=1}^{n'} a_{1,2k} & \forall \ (i_3, i_5, ..., i_{2n''+1}) \in \{0,1\}^{n''} \\ a_0 + a_1 - a_{2l} + \sum_{l \neq k=1}^{n'} a_{1,2k} + \sum_{j=2}^{n} a_j & l = 1, ..., n' \end{cases}$$

are non-negative and each of the $2^n - (2^{n''} + n')$ remaining ones can take negative values.

For example, consider the Hermitian operator

$$\mathcal{W}^{(2)}_{GHZ} = a_0 I_4 + a_1 S^{(\mathrm{GHZ})}_1 + a_2 S^{(\mathrm{GHZ})}_2 + a_{1,2} S^{(\mathrm{GHZ})}_1 S^{(\mathrm{GHZ})}_2 \tag{3.15}$$

with the following eigenvalues

$$\begin{aligned} \omega_1 = a_0 + a_1 + a_2 + a_{1,2} \quad &, \quad \omega_2 = a_0 + a_1 - a_2 - a_{1,2} \\ \omega_3 = a_0 - a_1 + a_2 - a_{1,2} \quad &, \quad \omega_4 = a_0 - a_1 - a_2 + a_{1,2} \ . \end{aligned} \tag{3.16}$$

We need only to consider the product state $|x^+\rangle|x^+\rangle$ corresponding to the vertex point $(1,0,0)$ since the product states corresponding to the other vertex points can be obtained by applying the Clifford operations $H \otimes H$, $M \otimes M$ and $\sigma_z \otimes I$ on this product state. Putting the vertex points in $Tr(\mathcal{W}^{(2)}_{GHZ}|\Upsilon\rangle\langle\Upsilon|) \geq 0$ yields

$$a_0 \geq |a_1|, \quad a_0 \geq |a_2|, \quad a_0 \geq |a_{12}| \ .$$

So in the parameters space, the allowed values of a's lie inside a cube with edge length a_0. The intersection of half-spaces $\omega_i \geq 0$ $(i = 1, .., 4)$ is a polyhedron inside the cube whose vertices coincide with four vertices of the cube and contains just the positive operators; the remaining part of the cube is the region of SEWs. On the other hand the variables P_i lie in the interval $[-1, 1]$ and form a cube in the space of variables. The convex hull of vertex points lies inside this cube and has the eight boundary half-spaces

$$|P_1 \pm P_2 + P_{1,2}| \leq 1 \quad , \quad |P_1 \pm P_2 - P_{1,2}| \leq 1 \ . \tag{3.17}$$

The above half-spaces define the feasible region (see Fig.1). Four of these half-spaces which correspond to the positive operators

$$\begin{aligned} {}^1\mathcal{P}_{GHZ} &= \left(I_4 + S^{(\mathrm{GHZ})}_1 + S^{(\mathrm{GHZ})}_2 + S^{(\mathrm{GHZ})}_{12}\right) = 4\ |\psi_{00}\rangle\langle\psi_{00}| \\ {}^2\mathcal{P}_{GHZ} &= \sigma^{(1)}_z ({}^1\mathcal{P}_{GHZ})\sigma^{(1)}_z = I_4 - S^{(\mathrm{GHZ})}_1 + S^{(\mathrm{GHZ})}_2 - S^{(\mathrm{GHZ})}_{12} \\ {}^3\mathcal{P}_{GHZ} &= \sigma^{(1)}_x ({}^1\mathcal{P}_{GHZ})\sigma^{(1)}_x = I_4 + S^{(\mathrm{GHZ})}_1 - S^{(\mathrm{GHZ})}_2 - S^{(\mathrm{GHZ})}_{12} \\ {}^4\mathcal{P}_{GHZ} &= \sigma^{(1)}_y ({}^1\mathcal{P}_{GHZ})\sigma^{(1)}_y = I_4 - S^{(\mathrm{GHZ})}_1 - S^{(\mathrm{GHZ})}_2 + S^{(\mathrm{GHZ})}_{12} \end{aligned} \tag{3.18}$$

are in one-one correspondence with four vertices of the cube in parameter space which are the same as the vertices of polyhedron formed by the positive operators.

For the purpose of later use, we introduce

$$|\psi_{i_1 i_2 \dots i_n}\rangle = (\sigma_z)^{i_1} \otimes (\sigma_x)^{i_2} \otimes \dots \otimes (\sigma_x)^{i_n} |\psi_{00\dots 0}\rangle, \tag{3.19}$$

where $|\psi_{00\dots 0}\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$ is the n-qubit GHZ state. As implied by the Eq. (3.18), the three last positive operators can be obtained from the first one via the action of some operations of the Clifford group. The other four boundary half-spaces which correspond to the optimal d-EWs

$$\begin{aligned}
{}^1\mathcal{W}^{(opt)}_{GHZ} &= I_4 - S^{(\mathrm{GHZ})}_1 - S^{(\mathrm{GHZ})}_2 - S^{(\mathrm{GHZ})}_{12} = 4(|\psi_{11}\rangle\langle\psi_{11}|)^{T_1} \\
{}^2\mathcal{W}^{(opt)}_{GHZ} &= \sigma^{(1)}_x({}^1\mathcal{W}^{(opt)}_{GHZ})\sigma^{(1)}_x = I_4 - S^{(\mathrm{GHZ})}_1 + S^{(\mathrm{GHZ})}_2 + S^{(\mathrm{GHZ})}_{12} \\
{}^3\mathcal{W}^{(opt)}_{GHZ} &= \sigma^{(1)}_z({}^1\mathcal{W}^{(opt)}_{GHZ})\sigma^{(1)}_z = I_4 + S^{(\mathrm{GHZ})}_1 - S^{(\mathrm{GHZ})}_2 + S^{(\mathrm{GHZ})}_{12} \\
{}^4\mathcal{W}^{(opt)}_{GHZ} &= \sigma^{(1)}_y({}^1\mathcal{W}^{(opt)}_{GHZ})\sigma^{(1)}_y = I_4 + S^{(\mathrm{GHZ})}_1 + S^{(\mathrm{GHZ})}_2 - S^{(\mathrm{GHZ})}_{12}
\end{aligned} \tag{3.20}$$

are in one-one correspondence with the remaining four vertices of the cube in parameters space. From Eq. (3.20) we see that the three last optimal d-EWs can be also obtained from the first one via the action of some operations of the Clifford group. So as we had in [19], the operators corresponding to the boundary planes are either optimal SEWs or positive operators. In this case, all of the witnesses are d-EWs since we can write them as a convex combination of an optimal d-EW and a positive operator from its opposite positive boundary plane.

3.2 Multi-qubit cluster EWs

We continue with EWs which can be constructed by the stabilizer operators of the cluster state and again consider two even and odd cases of the cluster SEWs which lie in the realm of exact LP problems (refer to appendix III for odd case).

3.2.1 Even case

Let us consider the following Hermitian operators

$$\mathcal{W}_{C}^{(n)}=a_{0}I_{2^{n}}+\sum_{k=1}^{n'}a_{2k}S_{2k}^{(\mathrm{C})}+a_{2m-1}S_{2m-1}^{(\mathrm{C})}+a_{2m-1,2m}S_{2m-1}^{(\mathrm{C})}S_{2m}^{(\mathrm{C})},\quad m=2,...,\left[\frac{n+1}{2}\right]-1 \quad (3.21)$$

In addition to the above operators, one can consider other Hermitian operators which differ from the above operators only in the last terms, that is the last terms of them are $a_{2m-2,2m-1}S_{2m-2}^{(\mathrm{C})}S_{2m-1}^{(\mathrm{C})}$ with $m=2,...,\left[\frac{n+1}{2}\right]$. However, we will consider only the operators (3.21) since the treatment is the same for others. Due to the commutativity of all cluster stabilizer generators, it is easy to see that the eigenvalues of $\mathcal{W}_{C}^{(n)}$ are

$$a_{0}+\sum_{j=1}^{n'}(-1)^{i_{2j}}a_{2j}+(-1)^{i_{2m-1}}a_{2m-1}+(-1)^{i_{2m-1}+i_{2m}}a_{2m-1,2m}\quad,\quad\forall\,(i_{1},i_{2},...,i_{n})\in\{0,1\}^{n} \quad (3.22)$$

To reduce the problem to a LP one and determine the feasible region, we require to know the vertices, namely the extreme points of the feasible region. For a separable state ρ_{s}, the non-negativity of

$$Tr(\mathcal{W}_{C}^{(n)}\rho_{s})\geq 0$$

implies the non-negativity of the objective function

$$\mathcal{F}_{\mathcal{W}_{C}^{(n)}}=a_{0}+\sum_{k=1}^{n'}a_{2k}P_{2k}+a_{2m-1}P_{2m-1}+a_{2m-1,2m}P_{2m-1,2m}\quad,\quad m=2,...,\left[\frac{n+1}{2}\right]-1 \quad (3.23)$$

where,

$$P_{2k}=Tr(S_{2k}^{(\mathrm{C})}\rho_{s})\quad,\quad P_{2m-1,2m}=Tr(S_{2m-1}^{(\mathrm{C})}S_{2m}^{(\mathrm{C})}\rho_{s}),$$

and all of the P_{2k}'s and $P_{2m-1,2m}$'s lie in the interval $[-1,1]$. The product vectors and the vertex points of the feasible region coming from these product vectors are listed in table 2

where

$$|\Phi\rangle=|z^{+}\rangle_{1}|x^{+}\rangle_{2}|z^{+}\rangle_{3}|x^{+}\rangle_{4}|z^{+}\rangle_{5}...|x^{+}\rangle_{n-1}|z^{+}\rangle_{n}$$

$$\Lambda_{i_{1},...,i_{n'}}^{(ev)}=\bigotimes_{j=1}^{n'}\left(\sigma_{z}^{(2j)}\right)^{i_{j}}\quad,\quad\forall\,(i_{1},i_{2},...,i_{n'})\in\{0,1\}^{n'}$$

$$\Lambda'^{(ev)}_{i_{1},...,i_{n'}}=\Lambda_{i_{1},...,i_{n'}}^{(ev)}H^{(2m-2)}H^{(2m-1)}H^{(2m)}$$

$$\Lambda''^{(ev)}_{i_{1},...,i_{n'}}=\Lambda_{i_{1},...,i_{n'}}^{(ev)}H^{(2m-2)}M^{(2m-1)}H^{(2m-1)}M^{(2m)}$$

Product state	$(P_2, P_4, ..., P_{2m-4}, P_{2m-2}, P_{2m-1}, P_{2m}, P_{2m+2}, ..., P_{2n'}, P_{2m-1,2m})$
$\Lambda^{(ev)}_{i_1,...,i_{n'}}\lvert\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, (-1)^{i_{m-1}}, 0, (-1)^{i_m}, (-1)^{i_{m+1}}, ..., (-1)^{i_{n'}}, 0\big)$
$\Lambda'^{(ev)}_{i_1,...,i_{n'}}\lvert\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, \pm 1, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n'}}, 0\big)$
$\Lambda''^{(ev)}_{i_1,...,i_{n'}}\lvert\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, 0, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n'}}, (-1)^{i_m}\big)$

Table 2: The product vectors and coordinates of vertices for $\mathcal{W}^{(n)}_{C}$.

For a given m, the convex hull of the above vertices, the feasible region, is a $(2n'+12)$-simplex formed by the intersection of the following half-spaces

$$\begin{gathered}
|P_{2m-1} \pm P_{2m-2} + P_{2m-1,2m}| \leq 1 \\
|P_{2m-1} \pm P_{2m-2} - P_{2m-1,2m}| \leq 1 \\
|P_{2m-1} \pm P_{2m} + P_{2m-1,2m}| \leq 1 \\
|P_{2m-1} \pm P_{2m} - P_{2m-1,2m}| \leq 1 \\
|P_{2k}| \leq 1 \quad , \quad m, m-1 \neq k = 1, ..., n'
\end{gathered} \tag{3.24}$$

(see Appendix II). In fact the half-spaces (3.24) come from the non-negativity of the expectation values of their corresponding operators

$$\begin{gathered}
I + S^{(\mathrm{C})}_{2m-1} \pm S^{(\mathrm{C})}_{2m-2} + S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \quad , \quad I - S^{(\mathrm{C})}_{2m-1} \mp S^{(\mathrm{C})}_{2m-2} - S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I + S^{(\mathrm{C})}_{2m-1} \pm S^{(\mathrm{C})}_{2m-2} - S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \quad , \quad I - S^{(\mathrm{C})}_{2m-1} \mp S^{(\mathrm{C})}_{2m-2} + S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I + S^{(\mathrm{C})}_{2m-1} \pm S^{(\mathrm{C})}_{2m} + S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \quad , \quad I - S^{(\mathrm{C})}_{2m-1} \mp S^{(\mathrm{C})}_{2m} - S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I + S^{(\mathrm{C})}_{2m-1} \pm S^{(\mathrm{C})}_{2m} - S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \quad , \quad I - S^{(\mathrm{C})}_{2m-1} \mp S^{(\mathrm{C})}_{2m} + S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I \pm S^{(\mathrm{C})}_{2k} \quad , \quad m, m-1 \neq k = 1, ..., n'
\end{gathered}$$

over pure product states. We note that it is not necessary to consider all the above operators, since one can obtain them just by applying some elements of the Clifford group on the 4

(compare with $2n' + 12$) following operators

$$
\begin{array}{c}
I \pm S^{(\mathrm{C})}_{2m-1} \pm S^{(\mathrm{C})}_{2m} \pm S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I - S^{(\mathrm{C})}_{2m-1} - S^{(\mathrm{C})}_{2m-2} - S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m} \\
I - S^{(\mathrm{C})}_{2}
\end{array}
\tag{3.25}
$$

For instance, the Clifford operation

$$U = (CN_{42})(CN_{53})(CN_{13})(CN_{24}) \in Cl(n)$$

transforms $S^{(\mathrm{C})}_{2}$ to $S^{(\mathrm{C})}_{4}$ by conjugation, i.e.,

$$U S^{(\mathrm{C})}_{2} U^{\dagger} = S^{(\mathrm{C})}_{4}$$

Now the problem of finding a pre-SEW of the form (3.21) is reduced to a LP problem with objective function (3.23) and constraints (3.24). If we put the coordinates of vertices (see table 2) in the objective function (3.23) and require the non-negativity of the objective function on all vertices we get the conditions

$$
\begin{array}{c}
a_0 \geq \sum_{j=1}^{n'} |a_{2j}| \\
a_0 \geq \sum_{j=1}^{m-2} |a_{2j}| + \sum_{j=m+1}^{n'} |a_{2j}| + |a_{2m-1}| \\
a_0 \geq \sum_{j=1}^{m-2} |a_{2j}| + \sum_{j=m+1}^{n'} |a_{2j}| + |a_{2m-1,2m}|
\end{array}
\tag{3.26}
$$

for the parameters a_i. Evidently, these conditions are sufficient to ensure that the objective function is non-negative on the whole of the feasible region. Cluster SEWs (3.21) and the odd case discussed in appendix III contain the SEWs in Eqs. (36) and (37) of Ref. [10] as special cases.

Fixing a_0 in the space of parameters, all of the a's lie inside the polygon defined by inequalities (3.26). Now in order that the operator of Eq.(3.21) becomes positive, all of its eigenvalues in (3.22) must be non-negative. The intersection of half-spaces arising from the non-negativity of eigenvalues form a polyhedron inside the aforementioned polygon. The same reasoning as in

the even case of GHZ SEWs, shows that the SEWs region is non-empty. For example, consider the operator

$$\mathcal{W}_{Cl}^{(4)} = a_0 I_{2^4} + a_1 S_1^{(\mathrm{C})} + a_2 S_2^{(\mathrm{C})} + a_4 S_4^{(\mathrm{C})} + a_{1,2} S_1^{(\mathrm{C})} S_2^{(\mathrm{C})}.$$

The eigenvalues of this operator are

$$\begin{array}{lll}
\omega_1 = a_0 + a_1 + a_2 + a_4 + a_{1,2} & , & \omega_2 = a_0 + a_1 - a_2 + a_4 - a_{1,2} \\
\omega_3 = a_0 + a_1 + a_2 - a_4 + a_{1,2} & , & \omega_4 = a_0 + a_1 - a_2 - a_4 - a_{1,2} \\
\omega_5 = a_0 - a_1 + a_2 + a_4 - a_{1,2} & , & \omega_6 = a_0 - a_1 + a_2 - a_4 - a_{1,2} \\
\omega_7 = a_0 - a_1 - a_2 + a_4 + a_{1,2} & , & \omega_8 = a_0 - a_1 - a_2 - a_4 + a_{1,2}
\end{array}$$

Without loss of generality we can assume that $a_1 \geq a_2$. With this assumption, the first four eigenvalues $\omega_1, \omega_2, \omega_3$ and ω_4 are always positive. Now let ω_5 and ω_6 be negative, i.e., $a_0 + a_2 < a_1 + a_{1,2}$. In this case, ω_7 and ω_8 can not be negative and vice versa. Therefore with these considerations, among the eight eigenvalues only the pair ω_5, ω_6 or ω_7, ω_8 can be negative. The explicit form of some four-qubit cluster SEWs is postponed to section 5.

4 Optimality of SEWs

Another advantage of stabilizer EWs is that the optimality of the EWs corresponding to the boundary hypereplanes of feasible region can be easily determined by a simple method presented here. Consider an EW corresponding to one of the hyper-planes in which three terms S_i, S_j and $S_i S_j$ appear simultaneously such as

$$\mathcal{W} = I + (-1)^{i_1} S_i + (-1)^{i_2} S_j + (-1)^{i_3} S_i S_j + \ldots \quad \forall\, i_1, i_2, i_3, \ldots \in \{0, 1\}. \tag{4.27}$$

If there exist $\epsilon > 0$ and a positive operator $\mathcal{P} = |\psi\rangle\langle\psi|$, such that $\mathcal{W}' = \mathcal{W} - \epsilon|\psi\rangle\langle\psi|$ is again an EW then we conclude that $\mathcal{W}$ is not optimal, otherwise it is. Note that there is no restriction in taking $\mathcal{P}$ as a pure positive operator since every positive operator can be expressed as a sum of pure positive operators with positive coefficients, i.e., $\mathcal{P} = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with all $\lambda_i \geq 0$. If

$\mathcal{W}'$ be an EW, then $|\psi\rangle$ has to satisfy the constraint $Tr(|\psi\rangle\langle\psi|\Upsilon\rangle\langle\Upsilon|) = |\langle\psi|\Upsilon\rangle|^2 = 0$ for any pure product state $|\Upsilon\rangle$ satisfying $Tr(\mathcal{W}|\Upsilon\rangle\langle\Upsilon|) = 0$. In other words, $|\psi\rangle$ has to be orthogonal to all such pure product states.

Since in SEWs of the form (4.27) considered so far there is no pair of locally commuting operators, it is always possible to find pure product vectors $|\Upsilon\rangle$ for which one of the relations

$$\begin{aligned} S_i|\Upsilon\rangle &= (-1)^{i_1+1}|\Upsilon\rangle \\ S_j|\Upsilon\rangle &= (-1)^{i_2+1}|\Upsilon\rangle \\ S_iS_j|\Upsilon\rangle &= (-1)^{i_3+1}|\Upsilon\rangle \end{aligned} \tag{4.28}$$

hold. The expectation value of $\mathcal{W}$ over such $|\Upsilon\rangle$'s is zero and $|\psi\rangle$ cannot contain such pure product vectors. All the eigenvectors of a stabilizer operation can be chosen as pure product vectors, half of them with eigenvalue +1 and the other half with eigenvalue -1, such that the expectation value of other stabilizer operations over them be zero. Because of Hermiticity of stabilizer operations, their eigenvectors can be used as a basis.

Let us assume that $|\Upsilon_k\rangle$'s are pure product eigenvectors of S_i with eigenvalues $(-1)^{i_1+1}$ and $|\Upsilon_k^{\perp}\rangle$'s are its pure product eigenvectors with eigenvalues $(-1)^{i_1}$ that have been chosen according to the above prescription. So the expectation value of $\mathcal{W}$ over $|\Upsilon_k\rangle$'s is zero and $|\psi\rangle$ cannot contain $|\Upsilon_k\rangle$'s that is $|\psi\rangle = \sum_k |\Upsilon_k^{\perp}\rangle$. This implies that $S_i|\psi\rangle = (-1)^{i_1}|\psi\rangle$. By the same reasoning we conclude that $S_j|\psi\rangle = (-1)^{i_2}|\psi\rangle$ and $S_iS_j|\psi\rangle = (-1)^{i_3}|\psi\rangle$. On the other hand, we have $S_iS_j|\psi\rangle = (-1)^{i_2}S_i|\psi\rangle = (-1)^{i_1+i_2}|\psi\rangle$. Hence, if $i_3 \neq i_1 + i_2$, i.e., if $i_3 = i_1 + i_2 + 1$, we get into a contradiction and $\mathcal{W}$ is optimal. Therefore, among all SEWs of the form (4.27) the following ones are optimal

$$\mathcal{W}_{opt} = I + (-1)^{i_1}S_i + (-1)^{i_2}S_j + (-1)^{i_1+i_2+1}S_iS_j + ... \qquad \forall\ (i_1, i_2, ...) \in \{0,1\}^m. \tag{4.29}$$

With the same reasoning as above one can conclude that any SEW of the general form

$$\mathcal{W} = I + (-1)^{i_1}S_i + (-1)^{i_2}S_j + (-1)^{i_3}S_iS_k + ... \qquad \forall\ i_1, i_2, ... \in \{0,1\}. \tag{4.30}$$

with $j \neq k$ is not optimal.

For instance, in the case of the three-qubit GHZ-state,

$$\mathcal{W}^{(3)}_{GHZ} = a_0 I_8 + a_1 S_1^{(\mathrm{GHZ})} + a_2 S_2^{(\mathrm{GHZ})} + a_3 S_3^{(\mathrm{GHZ})} + a_{1,2} S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} \tag{4.31}$$

the boundary half-spaces of the feasible region are

$$|P_1 \pm P_j + P_{1,2}| \leq 1 \quad , \quad |P_1 \pm P_j - P_{1,2}| \leq 1 \qquad j = 2, 3 \tag{4.32}$$

Using Clifford group operations, we can obtain all of these half-spaces only from the three half-spaces

$$|P_1 + P_2 + P_{1,2}| \leq 1 \quad , \quad P_1 + P_3 + P_{1,2} \leq 1 \tag{4.33}$$

The operators corresponding to the above boundary half-spaces are

$$\begin{aligned}
\mathcal{Q}_{GHZ} &= I_8 + S_1^{(\mathrm{GHZ})} + S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} = 4\big(|\psi_{000}\rangle\langle\psi_{000}| + |\psi_{001}\rangle\langle\psi_{001}|\big) \\
{}^1\mathcal{W}_{GHZ} &= I_8 - S_1^{(\mathrm{GHZ})} - S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} = 4\big(|\psi_{110}\rangle\langle\psi_{110}| + |\psi_{111}\rangle\langle\psi_{111}|\big)^{T_2} \\
{}^2\mathcal{W}_{GHZ} &= I_8 - S_1^{(\mathrm{GHZ})} - S_3^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} = 4|\psi_{101}\rangle\langle\psi_{101}| + 4\big(|\psi_{110}\rangle\langle\psi_{110}|\big)^{T_2}.
\end{aligned} \tag{4.34}$$

It is seen that, in agreement with the above argument, ${}^1\mathcal{W}_{GHZ}$ is an optimal SEW but ${}^2\mathcal{W}_{GHZ}$ is not. Also, for the three-qubit cluster state,

$$\mathcal{W}'^{(3)}_{C} = a_0 I_8 + a_1 S_1^{(\mathrm{C})} + a_2 S_2^{(\mathrm{C})} + a_3 S_3^{(\mathrm{C})} + a_{1,2} S_1^{(\mathrm{C})} S_2^{(\mathrm{C})} \tag{4.35}$$

using Clifford group operations, we can obtain all of the boundary half-spaces only from the three half-spaces

$$|P_1 + P_2 + P_{1,2}| \leq 1 \quad , \quad P_2 + P_3 + P_{1,2} \leq 1$$

and the operators corresponding to the above boundary half-spaces are

$$\begin{aligned}
H^{(1)} H^{(3)} \mathcal{Q}_{GHZ} H^{(1)} H^{(3)} &= I_8 + S_1^{(\mathrm{C})} + S_2^{(\mathrm{C})} + S_1^{(\mathrm{C})} S_2^{(\mathrm{C})} \\
H^{(1)} H^{(3)\,1}\mathcal{W}_{GHZ} H^{(1)} H^{(3)} &= I_8 - S_1^{(\mathrm{C})} - S_2^{(\mathrm{C})} - S_1^{(\mathrm{C})} S_2^{(\mathrm{C})} \\
H^{(1)} H^{(3)\,2}\mathcal{W}_{GHZ} H^{(1)} H^{(3)} &= I_8 - S_2^{(\mathrm{C})} - S_3^{(\mathrm{C})} - S_1^{(\mathrm{C})} S_2^{(\mathrm{C})}
\end{aligned} \tag{4.36}$$

Clearly, local unitary operations U_{local} do not change the optimality of EWs under the conjugation action such as $U_{local} W_{op} U_{local}^{\dagger}$, hence among the above operators, the second one remains optimal while the third one remains non-optimal.

5 Decomposability of SEWs

Another interesting feature of EWs which is necessary to study about SEWs is decomposability. As it is well-known that every two-qubit EW is decomposable [8, 9, 35], we discuss the three-qubit systems or more.

5.1 Decomposability of $\mathcal{W}_{GHZ}^{(n)}$

First consider three-qubit GHZ SEWs. The inequalities (3.14) show that in the space of parameters all GHZ SEWs lie inside the hypercube (again by fixing a_0) but this statement does not mean that any point of the region inside the hypercube is an SEW. The region defined by the inequalities

$$a_0 + (-1)^{i_1} a_1 + (-1)^{i_2} a_2 + (-1)^{i_3} a_3 + (-1)^{i_1+i_2} a_{1,2} \geq 0 \quad (i_1, i_2, i_3) \in \{0,1\}^3 \tag{5.37}$$

is the place inside the hypercube where the operator $\mathcal{W}_{GHZ}^{(3)}$ have just positive eigenvalues and hence is positive. First we consider the decomposability or non-decomposability of SEWs lying over the edges of the hypercube. These SEWs come from ${}^1\mathcal{W}_{GHZ}$ and ${}^2\mathcal{W}_{GHZ}$ of (4.34) by Clifford operations. The ${}^1\mathcal{W}_{GHZ}$ and SEWs coming from it are optimal decomposable since their partial transpositions with respect to some particles are positive.

Now in the space of parameters a, we consider the coordinates of points as $(a_1, a_2, a_3, a_{1,2})$. Putting the following four points (which lie over the edges of hypercube) in $\mathcal{W}_{GHZ}^{(3)}$ gives the following optimal SEWs

$$\begin{aligned}
(1,1,0,-1) &\rightarrow I_8 + S_1^{(\mathrm{GHZ})} + S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} \\
(-1,1,0,1) &\rightarrow I_8 - S_1^{(\mathrm{GHZ})} + S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} \\
(1,-1,0,1) &\rightarrow I_8 + S_1^{(\mathrm{GHZ})} - S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})} \\
(-1,-1,0,1) &\rightarrow I_8 - S_1^{(\mathrm{GHZ})} - S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})} S_2^{(\mathrm{GHZ})}
\end{aligned} \tag{5.38}$$

The above SEWs are optimal decomposable since their partial transpositions with respect to some particles are positive . A convex cone which may be formed by connecting every four

points of Eq. (5.38) to its opposite positive hyper-plane in Eq. (5.37) is d-SEWs. Note that the remaining operators in Eq. (4.32) coming from some points in the space of parameters are either d-SEW or positive. Therefore we conclude that all the three-qubit GHZ stabilizer EWs are decomposable. The discussion for more than three-qubit is rather complicated. It is clear that every EW with positive partial transpose with respect to some particles is decomposable. Therefore imposing the condition

$$a_0 + \sum_{k=1}^{n}(-1)^{i_k} a_k + \sum_{k\in\mathcal{B}}(-1)^{i_1+i_k+1} a_{1,k} + \sum_{k\in\mathcal{A}\setminus\mathcal{B}}(-1)^{i_1+i_k} a_{1,k} \geq 0 \quad (i_1,...,i_n)\in\{0,1\}^n \tag{5.39}$$

which in turn implies $\mathcal{W}^{T_\mathcal{B}} \geq 0$, yields the GHZ decomposable SEWs where the $\mathcal{B}$ is any nonempty subset of the set $\mathcal{A}=\{2,4,...,2n'\}$. Here taking partial transpose with respect to the particles $2j$ and $2j-1$ with $j=1,...,n'$ leads to the same result.

In order to show that the $\mathcal{W}^{(n)}_{GHZ}$ for $n\geq 4$ contain some nd-EWs, we discuss the four-qubit case in detail. From (3.7), we have

$$\mathcal{W}^{(4)}_{GHZ} = a_0 I_{2^4} + \sum_{k=1}^{4} a_k S_k^{(\mathrm{GHZ})} + \sum_{k=1}^{2} a_{1,2k} S_1^{(\mathrm{GHZ})} S_{2k}^{(\mathrm{GHZ})}$$

Using the local Clifford operations, all the 48 Hermitian operators corresponding to boundary half-spaces of the feasible region can be obtained only from the following 5 ones

$$\begin{aligned}
{}^1\mathcal{W}^{(4)}_{GHZ} &= I_{16} + S_1^{(\mathrm{GHZ})} + S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})}\\
{}^2\mathcal{W}^{(4)}_{GHZ} &= I_{16} + S_1^{(\mathrm{GHZ})} + S_4^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})}\\
{}^3\mathcal{W}^{(4)}_{GHZ} &= I_{16} - S_1^{(\mathrm{GHZ})} - S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})}\\
{}^4\mathcal{W}^{(4)}_{GHZ} &= I_{16} - S_1^{(\mathrm{GHZ})} - S_4^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})}\\
{}^5\mathcal{W}^{(4)}_{GHZ} &= I_{16} - S_1^{(\mathrm{GHZ})} - S_3^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})}
\end{aligned} \tag{5.40}$$

Now consider the following density matrices

$$\rho_\pm = \frac{1}{16}\big[I_{16} \pm \frac{1}{2}(S_1^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})} - S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})})\big]. \tag{5.41}$$

One can easily check that $\rho_\pm$ are PPT entangled states and can be detected by the above SEWs, i.e.,

$$Tr({}^i\mathcal{W}^{(4)}_{GHZ}\rho_-) = -\frac{1}{2} \qquad \text{for} \quad i=1,2, \tag{5.42}$$

and

$$Tr({}^{i}\mathcal{W}_{GHZ}^{(4)}\rho_{+}) = -\frac{1}{2} \qquad \text{for} \quad i = 3, 4, 5 \tag{5.43}$$

which means that all SEWs stated in Eq. (5.40) are nd-SEWs. On the other hand, by the (4.29), ${}^{3}\mathcal{W}_{GHZ}^{(4)}$ and ${}^{4}\mathcal{W}_{GHZ}^{(4)}$ are optimal SEWs.

Moreover, by the following transformations

$$\begin{gathered} {}^{i}\mathcal{W}_{GHZ}^{(4)} \longrightarrow {}^{i}\mathcal{W}' = U_{local}\ {}^{i}\mathcal{W}_{GHZ}^{(4)} U_{local}^{\dagger} \\ \rho_{\pm} \longrightarrow \rho'_{\pm} = U_{local}\rho_{\pm}U_{local}^{\dagger} \end{gathered} \tag{5.44}$$

where U_{local} may be any local unitary Clifford operation we can get the new nd-SEWs ${}^{i}\mathcal{W}'$ which can detect the PPT entangled states $\rho'_{\pm}$. It is necessary to mention that local unitary operations transform a PPT entangled state to a PPT one.

5.2 Decomposability of $\mathcal{W}_{C}^{(n)}$

Since the three-qubit cluster SEWs are transformed to three-qubit GHZ SEWs by local unitary Clifford operations as in Eq. (4.36) therefore they are also d-SEWs. For more than three-qubit the discussion is similar to the GHZ one. The SEWs

$$I_{2^n} - S_{2m-1}^{(\mathrm{C})} - S_{2m}^{(\mathrm{C})} - S_{2m-1}^{(\mathrm{C})} S_{2m}^{(\mathrm{C})} \tag{5.45}$$

are optimal d-SEWs since they have positive partial transpose with respect to the particle $2m-1$ or $2m$. Again a convex cone which may be formed by connecting every points of Eq. (5.45) in the space of parameters to its opposite positive hyper-planes

$$a_0 + \sum_{j=1}^{n'} (-1)^{i_{2j}} a_{2j} + (-1)^{i_{2m-1}} a_{2m-1} + (-1)^{i_{2m-1}+i_{2m}} a_{2m-1,2m} \geq 0 \tag{5.46}$$

for all $(i_1, i_2, ..., i_n) \in \{0,1\}^n$, are d-SEWs.

For illustration, we discuss the odd case of 4-qubit cluster SEW in detail. From (III-6), we have

$$\mathcal{W'}_{C}^{(4)} = a_0 I_{2^4} + \sum_{k=0}^{1} a_{2k+1} S_{2k+1}^{(\mathrm{C})} + a_2 S_2^{(\mathrm{C})} + a_{2,3} S_2^{(\mathrm{C})} S_3^{(\mathrm{C})}$$

Using the local Clifford operations, all the 14 Hermitian operators corresponding to boundary half-spaces of the feasible region can be obtained only from the following 3 ones

$$\begin{aligned} {}^{1}\mathcal{W}'^{(4)}_{C} &= I_{2^4} + S^{(\mathrm{C})}_{2} + S^{(\mathrm{C})}_{3} + S^{(\mathrm{C})}_{2} S^{(\mathrm{C})}_{3}, \\ {}^{2}\mathcal{W}'^{(4)}_{C} &= I_{2^4} - S^{(\mathrm{C})}_{2} - S^{(\mathrm{C})}_{3} - S^{(\mathrm{C})}_{2} S^{(\mathrm{C})}_{3}, \\ {}^{3}\mathcal{W}'^{(4)}_{C} &= I_{2^4} - S^{(\mathrm{C})}_{1} - S^{(\mathrm{C})}_{2} - S^{(\mathrm{C})}_{2} S^{(\mathrm{C})}_{3}. \end{aligned} \tag{5.47}$$

Among the above operators, ${}^{1}\mathcal{W}'^{(4)}_{C}$ is positive since ${}^{1}\mathcal{W}'^{(4)}_{C} = (I + S^{(\mathrm{C})}_{2})(I + S^{(\mathrm{C})}_{3})$, and if we take partial transpose of the second one with respect to second particle we get

$$({}^{2}\mathcal{W}'^{(4)}_{C})^{T_2} = (I - S^{(\mathrm{C})}_{2})(I - S^{(\mathrm{C})}_{3}) \geq 0, \tag{5.48}$$

so ${}^{2}\mathcal{W}'^{(4)}_{C}$ is an optimal d-SEW.

Although we could not find bound entangled states which can be detected by exactly soluble cluster SEWs however we will be able to find such entangled states for approximately soluble cluster SEWs as discussed in section 7 and therefore we postpone to subsection 7.2 for more details.

6 Separable and Entangled stabilizer states

Once again consider the general form of operators which is the same as Eq. (2.3), i.e.,

$$\rho := \sum_{j_1, j_2, \ldots, j_{n-k}=0}^{1} b_{j_1, j_2, \ldots, j_{n-k}} S_1^{\,j_1} S_2^{\,j_2} \ldots S_{n-k}^{j_{n-k}} = c_0 I_{2^n} + \sum_{j \neq 0} c_j A_j \tag{6.49}$$

where for simplicity we have renamed the $S_1^{j_1} S_2^{\,j_2} \ldots S_{n-k}^{j_{n-k}}$ and $b_{j_1, j_2, \ldots, j_{n-k}}$ by A_j and c_j respectively. Positivity of ρ together with $b_{0,0,\ldots,0} = c_0 = \frac{1}{2^n}$ make (6.49) a density matrix. On the other hand, we assert that the conditions

$$\sum_{j_1, j_2, \ldots, j_m=0}^{1} |b_{j_1, j_2, \ldots, j_m}| \leq \frac{1}{2^{n-1}} \quad \text{or} \quad \sum_{j \neq 0} |c_j| \leq \frac{1}{2^n} \tag{6.50}$$

yields separable state. To see this we note that for any element A_j of $\mathcal{S}_{n-k}$ the operator $I + A_j$ is separable because it is the projection operator on the space spanned by the pure product

eigenvectors of A_j corresponding to the eigenvalues +1. So any convex combination of the operators $I + A_j$ such as

$$\varrho_{sep} := \frac{\mu}{2^n} I_{2^n} + \frac{(1-\mu)}{2^n} \sum_{j\neq 0} p_j (I_{2^n} + A_j) = \frac{I_{2^n}}{2^n} + \frac{(1-\mu)}{2^n} \sum_{j\neq 0} p_j A_j \tag{6.51}$$

is separable where $\sum_{j\neq 0} p_j = 1$ and $0 \leq \mu \leq 1$. The same statement holds if we replace some $I + A_j$ by $I - A_j$ in the above equation. Now if we consider all c_j to be positive in Eq. (6.49) and rename $\frac{(1-\mu)}{2^n} p_j$ by c_j (with $j \neq 0$) we conclude that the condition (6.50) is satisfied and therefore ρ is separable. For the cases that some c_j are negative it is enough to replace some $I + A_j$ by $I - A_j$ in the Eq. (6.51) and proceed the same way as described above. Consequently we get a family of separable states expressed in terms of the elements of the stabilizer group provided that the condition (6.50) satisfies. In the following, some entangled states including PPT ones which can be detected by GHZ and cluster SEWs are introduced.

6.1 Entangled states which can be detected by $\mathcal{W}^{(n)}_{GHZ}$

Now we assert that GHZ stabilizer EWs can detect some mixed density matrices. To this aim consider the following operator

$$\rho^{(n)}_{GHZ} := \sum_{j_1, j_2, ..., j_n = 0}^{1} b_{j_1, j_2, ..., j_n} S_1^{(\mathrm{GHZ})^{j_1}} S_2^{(\mathrm{GHZ})^{j_2}} ... S_n^{(\mathrm{GHZ})^{j_n}} \tag{6.52}$$

which due to tracelessness of $S_i^{(\mathrm{GHZ})}$ the condition $Tr(\rho^{(n)}_{GHZ}) = 1$ gives $b_{0,0,...,0} = \frac{1}{2^n}$ and the positivity of density matrix impose

$$\sum_{j_1, j_2, ..., j_n = 0}^{1} (-1)^{i_1 j_1 + i_2 j_2 + ... + i_n j_n} \; b_{j_1, j_2, ..., j_n} \geq 0 \quad , \quad \forall \; (i_1, i_2, ..., i_n) \in \{0, 1\}^n \tag{6.53}$$

to its eigenvalues. An interesting case is when all coefficients are equal to $b_{j_1, j_2, ..., j_n} = \frac{1}{2^n}$ which is coincides with the n-qubit GHZ state

$$|\psi_{00...0}\rangle\langle\psi_{00...0}| = \frac{1}{2^n} \prod_{j=1}^{n} (I + S_j^{(\mathrm{GHZ})}).$$

This density matrix has 2^n terms which except $b_{0,0,...,0}$ the other are arbitrary parameters with the constraints in Eq. (6.55). These 2^n constraints forms a simplex polygon in a $2^n - 1$ dimensional space with coordinate variables $b_{j_1,j_2,...,j_n}$ (excepted $b_{0,0,...,0}$). Furthermore if we want $\rho^{(n)}_{GHZ}$ becomes a PPT entangled state in the sense that its partial transpose is positive definite with respect to any particle, i.e., ${\rho^{(n)}_{GHZ}}^{T_i} \geq 0$ with $i = 1, ..., n$ then we must have

$$\sum_{j_1,j_2,...,j_n=0}^{1} \{(-1)^{i_1}\ b_{1,j_2,...,j_n} + (-1)^{i_2 j_2 + i_3 j_3 + ... + i_n j_n}\ b_{0,j_2,...,j_n}\} \geq 0 \quad , \forall\ (i_1, i_2, ..., i_n) \in \{0,1\}^n$$

Introducing the new parameters $b_i = b_{0,...,0,1,0,...,0}$ with a 1 in the ith position, and $b_{1,j} = b_{1,0,...,0,1,0,...,0}$ with a 1 in the jth position , and using the orthogonality (I-2) of S_i's, then the condition for detectability of $\rho^{(n)}_{GHZ}$ by $\mathcal{W}^{(n)}_{GHZ}$ can be written as

$$Tr(\mathcal{W}^{(n)}_{GHZ} \rho^{(n)}_{GHZ}) = \frac{a_0}{2^n} + \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n'} a_{1,2k} b_{1,2k} < 0$$

6.2 Entangled states which can be detected by $\mathcal{W}^{(n)}_{C}$

Now we assert that the above cluster stabilizer EWs can detect some mixed density matrices. To this aim consider the following operator

$$\rho^{(n)}_{C} := \sum_{j_1,j_2,...,j_n=0}^{1} b_{j_1,j_2,...,j_n} S_1^{(\mathrm{C})j_1} S_2^{(\mathrm{C})j_2} ... S_n^{(\mathrm{C})j_n} \tag{6.54}$$

which due to traceless of $S^{(\mathrm{C})}_i$ the condition $Tr(\rho) = 1$ gives $b_{0,0,...,0} = \frac{1}{2^n}$ and the positivity of density matrix impose

$$\sum_{j_1,j_2,...,j_n=0}^{1} (-1)^{i_1 j_1 + i_2 j_2 + ... + i_n j_n}\ b_{j_1,j_2,...,j_n} \geq 0 \quad , \quad \forall\ (i_1, i_2, ..., i_n) \in \{0,1\}^n \tag{6.55}$$

to the its eigenvalues. An interesting case is when all coefficients are equal to $b_{j_1,j_2,...,j_n} = \frac{1}{2^n}$ which is coincides with the n-qubit cluster state

$$|\mathrm{C}\rangle\langle\mathrm{C}| = \frac{1}{2^n} \prod_{j=1}^{n} (I + S^{(\mathrm{C})}_j).$$

In order to $\rho_C^{(n)}$ can be detected by an odd case $\mathcal{W}'^{(n)}_C$, we must have

$$Tr(\mathcal{W}'^{(n)}_C \rho_C^{(n)}) = \frac{a_0}{2^n} + \sum_{k=1}^{n} a_{2k+1} b_{2k+1} + a_{2m} b_{2m} + a_{2m,2m+1} b_{2m,2m+1} < 0.$$

7 Approximate stabilizer EWs

So far, we have considered SEWs which can be exactly solved by LP method. In this section, we consider approximately soluble SEWs which come from by adding some other members of stabilizer group to exactly soluble SEWs. In all of the SEWs discussed in section 3, the boundary half-spaces arise from the vertices which themselves come from pure product states and the resulting inequalities did not offend against the convex hull of vertices at all. But by adding some terms to exactly soluble SEWs, it may be happen that the feasible region be convex with curvature on some boundaries and the problem can not be solved by exactly LP method. In these cases the linear constraints no longer arise from convex hull of the vertices coming from pure product states. Hence we transform such problem to approximately soluble LP one. Our approach is to draw the hyper-planes tangent to feasible region and parallel to hyper-planes coming from vertices and in this way we enclose the feasible regions by such hyper-planes. It is clear that in this extension, the vertices no longer arise from pure product states.

7.1 Approximate n-qubit GHZ SEWs

For the even case of GHZ SEWs we add one of the statements $S_1^{(\mathrm{GHZ})} S_{2l+1}^{(\mathrm{GHZ})}$ $(l = 1, ..., n'')$ to Eq. (3.7) as

$$\mathcal{W}^{(n)}_{GHZ_{(ap)}} = a_0 I_{2^n} + \sum_{k=1}^{n} a_k S_k^{(\mathrm{GHZ})} + \sum_{k=1}^{n'} a_{1,2k} S_1^{(\mathrm{GHZ})} S_{2k}^{(\mathrm{GHZ})} + a_{1,2l+1} S_1^{(\mathrm{GHZ})} S_{2l+1}^{(\mathrm{GHZ})} \tag{7.56}$$

and try to solve it by the LP method. The eigenvalues of $\mathcal{W}^{(n)}_{GHZ_{(ap)}}$ are

$$a_{0} + \sum_{j=1}^{n}(-1)^{i_j}a_{j} + \sum_{k=1}^{n'}(-1)^{i_1+i_{2k}}a_{1,2k} + (-1)^{i_1+i_{2l+1}}a_{1,2l+1} \quad , \quad \forall\ (i_1, i_2, ..., i_n) \in \{0,1\}^n$$

The coordinates of the vertices which arise from pure product vectors are listed in the table 3

Product state	$(P_2, P_3, ..., P_{n-1}, P_n, P_1, P_{1,2}, P_{1,4}, ..., P_{1,2n'-2}, P_{1,2n'}, P_{1,2l+1})$
$\vert\Psi^{\pm}\rangle$	$(0, 0, ..., 0, 0, \pm1, 0, 0, ..., 0, 0, 0)$
$\Lambda_{1}\vert\Psi^{\pm}\rangle$	$(0, 0, ..., 0, 0, 0, \pm1, 0, ..., 0, 0, 0)$
$\vdots$	$\vdots$
$\Lambda_{n'}\vert\Psi^{\pm}\rangle$	$(0, 0, ..., 0, 0, 0, 0, 0, ..., 0, \pm1, 0)$
$\Lambda^{(\mathrm{ap})}_{2l+1}\vert\Psi^{\pm}\rangle$	$(0, 0, ..., 0, 0, 0, 0, 0, ..., 0, 0, \pm1)$
$\Xi_{i_2,...,i_n}\vert\Psi^{+}\rangle$	$\big((-1)^{i_2}, (-1)^{i_2+i_3}, ..., (-1)^{i_{n-2}+i_{n-1}}, (-1)^{i_{n-1}+i_n}, 0, 0, 0, ..., 0, 0, 0\big)$

Table 3: The product vectors which seem to be the vertices for $\mathcal{W}^{(n)}_{GHZ_{(ap)}}$.

where

$$\Lambda^{(\mathrm{ap})}_{2l+1} = \left(M^{(2l)}\right)^{\dagger} M^{(2l+1)}.$$

Choosing any $N_1 = n + n' + 1$ points among $N_2 = 2^{n-1} + n' + 2$ above points we get the following $\mathbf{C}^{N_2}_{N_1}$ half-spaces

$$|P_{1} \pm P_{j} + \sum_{k=1}^{n'}(-1)^{i_k}P_{1,2k} + (-1)^{i_{n'+1}}P_{1,2l+1}| \leq \mu_{max} =?, \qquad j = 2, ..., n, \tag{7.57}$$

where $(i_1, ..., i_{n'+1}) \in \{0,1\}^{n'+1}$. But calculations show that the inequalities offend against 1 up to $\mu_{max} = \frac{1+\sqrt{2}}{2}$ (see appendix II). This shows that the problem does not lie in the realm of exactly soluble LP problems and we have to use approximate LP. To this aim, we shift aforementioned hyper-planes parallel to themselves such that they reach to maximum value $\mu_{max} = \frac{1+\sqrt{2}}{2}$. On the other hand the maximum shifting is where the hyper-planes become tangent to convex region coming from pure product states and in this manner we will be able

to encircle the feasible region by the half-spaces

$$\begin{aligned} |P_{1}+P_{2j}+\textstyle\sum_{k=1}^{n'}P_{1,2k}+P_{1,2l+1}| &\leq \tfrac{1+\sqrt{2}}{2} \qquad j=1,...,n' \\ |P_{1}+P_{2l+1}+\textstyle\sum_{k=1}^{n'}P_{1,2k}+P_{1,2l+1}| &\leq \tfrac{1+\sqrt{2}}{2} \\ P_{1}+P_{2j+1}+\textstyle\sum_{k=1}^{n'}P_{1,2k} &\leq \tfrac{1+\sqrt{2}}{2} \qquad l\neq j=1,...,n'' \\ P_{1} &\leq 1 \end{aligned} \tag{7.58}$$

where again we have used the Clifford group and write just the generating half-spaces. Due to the above inequalities the problem is reduced to the LP problem

$$\begin{aligned} &\text{minimize} \qquad Tr(\mathcal{W}^{(n)}_{GHZ_{(ap)}}|\gamma\rangle\langle\gamma|) \\ &\text{s.t.} \quad \begin{cases} |P_{1}\pm P_{j}+\sum_{k=1}^{n'}(-1)^{i_k}P_{1,2k}+(-1)^{i_{n'+1}}P_{1,2l+1}|\leq \frac{1+\sqrt{2}}{2} & j=2,...,n \\ P_{i}\leq 1 \qquad i=1,...,n \\ P_{1,2k}\leq 1 \qquad k=1,...,n' \\ P_{1,2l+1}\leq 1 \end{cases} \end{aligned}$$

for all $(i_1,i_2,...,i_{n'},i_{n'+1})\in\{0,1\}^{n'+1}$, where it can be solved by simplex method.

The intersections of the half-spaces in the above equation form a convex polygon whose vertices lie at any permutation P'_{1},P'_{j}, $P'_{1,2k}(k=1,...,n')$ and $P'_{1,2l+1}$ with a given j $(j=2,...,n)$ of the points listed in table 4 where P' 's are defined by shifting the P 's for all $(i_1,...,i_n,i_{1,2},...,i_{1,2n'})\in\{0,1\}^{n+n'}$.

So in order that the expectation value of $\mathcal{W}^{(n)}_{GHZ_{(ap)}}$ be non-negative over any pure product state, the following inequalities and any inequality obtained from them by permuting the parameters a_{1},a_{j}, $a_{1,2k}(k=1,...,n')$ and $a_{1,2l+1}$ with a given j for $j=2,...,n$, must be fulfilled

$$\begin{gathered} a_{0}+\textstyle\sum_{k=1}^{n}(-1)^{i_k}a_{k}+\sum_{k=1}^{n'}(-1)^{i_{1,2k}}a_{1,2k}+\frac{\sqrt{2}-3}{2}a_{1,2l+1}\geq 0 \\ \text{such that } \ (-1)^{i_1}+(-1)^{i_j}+\textstyle\sum_{k=1}^{n'}(-1)^{i_{1,2k}}=2 \end{gathered}$$

$$\begin{gathered} a_{0}+\textstyle\sum_{k=1}^{n}(-1)^{i_k}a_{k}+\sum_{k=1}^{n'}(-1)^{i_{1,2k}}a_{1,2k}+\frac{3-\sqrt{2}}{2}a_{1,2l+1}\geq 0 \\ \text{such that } \ (-1)^{i_1}+(-1)^{i_j}+\textstyle\sum_{k=1}^{n'}(-1)^{i_{1,2k}}=-2 \end{gathered}$$

$(P'_2, P'_3, ..., P'_{n-1}, P'_n, P'_1, P'_{1,2}, P'_{1,4}, ..., P'_{1,2n'-2}, P'_{1,2n'}, P'_{1,2l+1})$
$\big((-1)^{i_2}, (-1)^{i_3}, ..., (-1)^{i_{n-1}}, (-1)^{i_n}, (-1)^{i_1}, (-1)^{i_{1,2}}, (-1)^{i_{1,4}}, ..., (-1)^{i_{1,2n'-2}}, (-1)^{i_{1,2n'}}, \frac{\sqrt{2}-3}{2}\big)$ $\ni P'_1 + P'_j + \sum_{k=1}^{n'} P'_{1,2k} = 2$
$\big((-1)^{i_2}, (-1)^{i_3}, ..., (-1)^{i_{n-1}}, (-1)^{i_n}, (-1)^{i_1}, (-1)^{i_{1,2}}, (-1)^{i_{1,4}}, ..., (-1)^{i_{1,2n'-2}}, (-1)^{i_{1,2n'}}, \frac{3-\sqrt{2}}{2}\big)$ $\ni P'_1 + P'_j + \sum_{k=1}^{n'} P'_{1,2k} = -2$
$\big((-1)^{i_2}, (-1)^{i_3}, ..., (-1)^{i_{n-1}}, (-1)^{i_n}, (-1)^{i_1}, (-1)^{i_{1,2}}, (-1)^{i_{1,4}}, ..., (-1)^{i_{1,2n'-2}}, (-1)^{i_{1,2n'}}, \frac{\sqrt{2}-1}{2}\big)$ $\ni P'_1 + P'_j + \sum_{k=1}^{n'} P'_{1,2k} = 1$
$\big((-1)^{i_2}, (-1)^{i_3}, ..., (-1)^{i_{n-1}}, (-1)^{i_n}, (-1)^{i_1}, (-1)^{i_{1,2}}, (-1)^{i_{1,4}}, ..., (-1)^{i_{1,2n'-2}}, (-1)^{i_{1,2n'}}, \frac{1-\sqrt{2}}{2}\big)$ $\ni P'_1 + P'_j + \sum_{k=1}^{n'} P'_{1,2k} = -1$

Table 4: The coordinates of vertices for $\mathcal{W}^{(n)}_{GHZ_{(ap)}}$.

$$\begin{array}{c} a_0 + \sum_{k=1}^{n}(-1)^{i_k}a_k + \sum_{k=1}^{n'}(-1)^{i_{1,2k}}a_{1,2k} + \frac{\sqrt{2}-1}{2}a_{1,2l+1} \geq 0 \\ \text{such that } \ (-1)^{i_1} + (-1)^{i_j} + \sum_{k=1}^{n'}(-1)^{i_{1,2k}} = 1 \\ \\ a_0 + \sum_{k=1}^{n}(-1)^{i_k}a_k + \sum_{k=1}^{n'}(-1)^{i_{1,2k}}a_{1,2k} + \frac{1-\sqrt{2}}{2}a_{1,2l+1} \geq 0 \\ \text{such that } \ (-1)^{i_1} + (-1)^{i_j} + \sum_{k=1}^{n'}(-1)^{i_{1,2k}} = -1 \end{array} \tag{7.59}$$

Similarly, one could repeat this approximation for the odd case of GHZ SEWs like above.

As the case of exactly soluble GHZ SEWs, we assert that there exist some nd-SEWs among the approximately soluble GHZ SEWs. To see this, consider the four-qubit GHZ SEWs

$$\mathcal{W}_{\pm} = \frac{1+\sqrt{2}}{2}\, I_{16} + S_1^{(\mathrm{GHZ})} + S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_2^{(\mathrm{GHZ})} + S_1^{(\mathrm{GHZ})}S_4^{(\mathrm{GHZ})} \pm S_1^{(\mathrm{GHZ})}S_3^{(\mathrm{GHZ})} \tag{7.60}$$

which both can detect the PPT entangled state in Eq. (5.41) with $Tr(\mathcal{W}_{\pm}\rho_{-}) = -\frac{2-\sqrt{2}}{2} \simeq -0.29$.

7.2 Approximated n-qubit Cluster SEWs

For the odd case of cluster SEWs we add one of the statements $S^{(\mathrm{C})}_{2m-1}S^{(\mathrm{C})}_{2m}$ to Eq. (III-6) as

$$\mathcal{W}'^{(n)}_{C_{(ap.)}} = a_0 I_{2^n} + \sum_{k=0}^{n''} a_{2k+1} S^{(\mathrm{C})}_{2k+1} + a_{2m} S^{(\mathrm{C})}_{2m} + a_{2m,2m+1} S^{(\mathrm{C})}_{2m} S^{(\mathrm{C})}_{2m+1} + a_{2m-1,2m} S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m}, \quad (7.61)$$

where $m = 1, ..., n''$. The eigenvalues of $\mathcal{W}'^{(n)}_{C_{(ap.)}}$ for all $(i_1, i_2, ..., i_n) \in \{0,1\}^n$ are

$$a_0 + \sum_{j=0}^{n''} (-1)^{i_{2j+1}} a_{2j+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m}+i_{2m+1}} a_{2m,2m+1} + (-1)^{i_{2m-1}+i_{2m}} a_{2m-1,2m}$$

The coordinates of the vertices which arise from pure product vectors are listed in the Table 5, where

Product state	$(P_1, P_3, ..., P_{2m-3}, P_{2m-1}, P_{2m}, P_{2m+1}, P_{2m+3}, ..., P_{2n''+1}, P_{2m,2m+1}, P_{2m-1,2m})$
$\Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}}\lvert\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, (-1)^{i_{m-1}}, 0, (-1)^{i_m}, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, 0, 0\big)$
$\Lambda'^{(odd)}_{i_1,i_2,...,i_{n''+1}}\lvert\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, \pm 1, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, 0, 0\big)$
$\Lambda''^{(odd)}_{i_1,i_2,...,i_{n''+1}}\lvert\Phi\rangle$	$((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, 0, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, (-1)^{i_m}, 0)$
$\Lambda'''^{(odd)}_{i_1,i_2,...,i_{n''+1}}\lvert\Phi\rangle$	$((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, 0, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, 0, (-1)^{i_m})$

Table 5: The product vectors which seem to be the vertices for $\mathcal{W}'^{(n)}_{C_{(ap)}}$.

$$\Lambda'''^{(odd)}_{i_1,i_2,...,i_{n''+1}} = \Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}} H^{(2m-2)} M^{(2m-1)} H^{(2m-1)} M^{(2m)}$$

By choosing any $n''+4$ from $2^{n''+1} + 3 \times 2^{n''}$ points, the following half-spaces achieves

$$\begin{array}{l} |P_{2m} + (-1)^{i_1} P_{2m-1} + (-1)^{i_2} P_{2m,2m+1} + (-1)^{i_3} P_{2m-1,2m}| \leq \frac{2}{\sqrt{3}} \\ |P_{2k+1}| \leq 1 \qquad , \qquad m-1 \neq k = 0, ..., n'' \end{array} \quad (7.62)$$

for all $(i_1, i_2, i_3) \in \{0,1\}^n$ (see appendix II). This shows that the problem does not lie in the realm of exact LP problems and we have to use approximate LP one. To do so, we shift

aforementioned hyper-planes parallel to themselves such that they reach to maximum value $\eta = \frac{2}{\sqrt{3}}$. On the other hand the maximum shifting is where the hyper-planes become tangent to convex region coming from pure product states and in this manner we will be able to encircle the feasible region by the half-spaces

$$\begin{gathered} |P_{2m} + P_{2m-1} + P_{2m,2m+1} + P_{2m-1,2m}| \leq \tfrac{2}{\sqrt{3}} \\ P_1 \leq 1 \end{gathered} \tag{7.63}$$

where again we have used the Clifford group and write just the generating half-spaces. Due to the above inequalities the problem is approximately reduced to

$$\begin{array}{ll} & \text{minimize} \quad Tr(\mathcal{W}'^{(n)}_{C_{(ap.)}} |\gamma\rangle\langle\gamma|) \\ \text{s.t.} & \begin{cases} |P_{2m} + (-1)^{i_1} P_{2m-1} + (-1)^{i_2} P_{2m,2m+1} + (-1)^{i_3} P_{2m-1,2m}| \leq \frac{2}{\sqrt{3}} & ,\forall\ (i_1, i_2, i_3) \in \{0,1\}^n \\ |P_{2m} + (-1)^{i_1} P_{2m+1} + (-1)^{i_2} P_{2m,2m+1} + (-1)^{i_3} P_{2m-1,2m}| \leq \frac{2}{\sqrt{3}} & ,\forall\ (i_1, i_2, i_3) \in \{0,1\}^n \\ |P_{2k+1}| \leq 1 \quad , \quad k = 0, ..., n'' \\ |P_{2m}| \leq 1 \\ |P_{2m,2m+1}| \leq 1 \\ |P_{2m-1,2m}| \leq 1 \end{cases} \end{array} \tag{7.64}$$

which can be solved by LP method.

The intersections of the half-spaces in the above equation form a convex polygon whose vertices lie at any permutation of the coordinates $P_{2m-1}, P_{2m}, P_{2m,2m+1}, P_{2m-1,2m}$ of the points listed in Table 12 and any permutation of the coordinates $P_{2m}, P_{2m+1}, P_{2m,2m+1}, P_{2m-1,2m}$ of the points listed in Table 13 where P' 's are defined by shifting the P 's and for all $(i_1, ..., i_n, i_{2m,2m+1}) \in \{0,1\}^{n+1}$.

Therefore, to be guaranteed the non-negativity of the expectation value of $\mathcal{W}'^{(n)}_{C_{(ap)}}$ over all pure product states, the conditions

$$\begin{gathered} a_0 + \textstyle\sum_{k=0}^{n''} (-1)^{i_{2k+1}} a_{2k+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m,2m+1}} a_{2m,2m+1} + \frac{2-\sqrt{3}}{\sqrt{3}} a_{2m-1,2m} \geq 0 \\ \text{such that} \ \ (-1)^{i_{2m-1}} + (-1)^{i_{2m}} + (-1)^{i_{2m-1,2m}} + (-1)^{i_{2m,2m+1}} = 1 \end{gathered}$$

$(P'_1, P'_3, ..., P'_{2m-3}, P'_{2m-1}, P'_{2m}, P'_{2m+1}, P'_{2m+3}, ..., P'_{2n''+1}, P'_{2m,2m+1}, P'_{2m-1,2m})$
$\big((-1)^{i_1}, (-1)^{i_3}, ..., (-1)^{i_{2m-3}}, (-1)^{i_{2m-1}}, (-1)^{i_{2m}}, (-1)^{i_{2m+1}}, (-1)^{i_{2m+3}}, ..., (-1)^{i_{2n''+1}}, (-1)^{i_{2m,2m+1}}, \frac{2-\sqrt{3}}{\sqrt{3}}\big)$ $\ni P'_{2m} + P'_{2m-1} + P'_{2m,2m+1} + P'_{2m-1,2m} = 1$
$\big((-1)^{i_1}, (-1)^{i_3}, ..., (-1)^{i_{2m-3}}, (-1)^{i_{2m-1}}, (-1)^{i_{2m}}, (-1)^{i_{2m+1}}, (-1)^{i_{2m+3}}, ..., (-1)^{i_{2n''+1}}, (-1)^{i_{2m,2m+1}}, \frac{\sqrt{3}-2}{\sqrt{3}}\big)$ $\ni P'_{2m} + P'_{2m-1} + P'_{2m,2m+1} + P'_{2m-1,2m} = -1$

Table 6: The coordinates of vertices for $\mathcal{W'}^{(n)}_{C_{(ap)}}$.

$(P'_1, P'_3, ..., P'_{2m-3}, P'_{2m-1}, P'_{2m}, P'_{2m+1}, P'_{2m+3}, ..., P'_{2n''+1}, P'_{2m,2m+1}, P'_{2m-1,2m})$
$\big((-1)^{i_1}, (-1)^{i_3}, ..., (-1)^{i_{2m-3}}, (-1)^{i_{2m-1}}, (-1)^{i_{2m}}, (-1)^{i_{2m+1}}, (-1)^{i_{2m+3}}, ..., (-1)^{i_{2n''+1}}, (-1)^{i_{2m,2m+1}}, \frac{2-\sqrt{3}}{\sqrt{3}}\big)$ $\ni P'_{2m} + P'_{2m+1} + P'_{2m,2m+1} + P'_{2m-1,2m} = 1$
$\big((-1)^{i_1}, (-1)^{i_3}, ..., (-1)^{i_{2m-3}}, (-1)^{i_{2m-1}}, (-1)^{i_{2m}}, (-1)^{i_{2m+1}}, (-1)^{i_{2m+3}}, ..., (-1)^{i_{2n''+1}}, (-1)^{i_{2m,2m+1}}, \frac{\sqrt{3}-2}{\sqrt{3}}\big)$ $\ni P'_{2m} + P'_{2m+1} + P'_{2m,2m+1} + P'_{2m-1,2m} = -1$

Table 7: The vertex points of approximated FR of $\mathcal{W'}^{(n)}_{C_{(ap)}}$.

$$\begin{gathered} a_0 + \textstyle\sum_{k=0}^{n''}(-1)^{i_{2k+1}} a_{2k+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m,2m+1}} a_{2m,2m+1} + \frac{\sqrt{3}-2}{\sqrt{3}} a_{2m-1,2m} \geq 0 \\ \text{such that } \ (-1)^{i_{2m-1}} + (-1)^{i_{2m}} + (-1)^{i_{2m-1,2m}} + (-1)^{i_{2m,2m+1}} = -1 \end{gathered} \tag{7.65}$$

and any permutation of parameters $a_{2m-1}, a_{2m}, a_{2m,2m+1}$ and $a_{2m-1,2m}$ together with the following conditions

$$\begin{gathered} a_0 + \textstyle\sum_{k=0}^{n''}(-1)^{i_{2k+1}} a_{2k+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m,2m+1}} a_{2m,2m+1} + \frac{2-\sqrt{3}}{\sqrt{3}} a_{2m-1,2m} \geq 0 \\ \text{such that } \ (-1)^{i_{2m-1}} + (-1)^{i_{2m}} + (-1)^{i_{2m-1,2m}} + (-1)^{i_{2m,2m+1}} = 1 \end{gathered}$$

$$\begin{gathered} a_0 + \textstyle\sum_{k=0}^{n''}(-1)^{i_{2k+1}} a_{2k+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m,2m+1}} a_{2m,2m+1} + \frac{\sqrt{3}-2}{\sqrt{3}} a_{2m-1,2m} \geq 0 \\ \text{such that } \ (-1)^{i_{2m-1}} + (-1)^{i_{2m}} + (-1)^{i_{2m-1,2m}} + (-1)^{i_{2m,2m+1}} = -1 \end{gathered} \tag{7.66}$$

and any permutation of parameters $a_{2m}, a_{2m+1}, a_{2m,2m+1}$ and $a_{2m-1,2m}$ must be fulfilled. Similarly, one could repeat this approximate solve for the even case of cluster SEWs just like above.

Now we discuss the non-decomposability of cluster SEWs mentioned at the end of subsection 5.2. For more than three-qubit, the SEWs corresponding to half-spaces (7.63) contain some nd-SEWs. As an instance, consider the SEW

$$\mathcal{W} = \frac{2}{\sqrt{3}}\, I_{16} - S_{1}^{(C)} - S_{2}^{(C)} - S_{1}^{(C)}S_{2}^{(C)} - S_{2}^{(C)}S_{3}^{(C)} \tag{7.67}$$

The expectation value of $\mathcal{W}$ with the following density matrix

$$\rho = \frac{1}{16}\big[I_{16} + \frac{1}{2}(S_{2}^{(C)} + S_{1}^{(C)}S_{2}^{(C)} + S_{2}^{(C)}S_{3}^{(C)} - S_{1}^{(C)}S_{2}^{(C)}S_{3}^{(C)})\big]$$

is $Tr(\mathcal{W}\rho) = -0.345$ which means that $\mathcal{W}$ can detect ρ. On the other hand one can easily check that ρ is a bound entangled state and hence $\mathcal{W}$ is a nd-SEW.

8 Conclusion

We have considered the construction of EWs by using the stabilizer operators of some given multi-qubit states. It was shown that when the feasible region is a polygon or can be approximated by a polygon, the problem is reduced to a LP one. For illustrating the method, several examples including GHZ, cluster, and exceptional states EWs have studied in details. The optimality and decomposability or non-decomposability of SEWs corresponding to boundary half-planes surrounding the feasible region have examined and it was shown that the optimality has a close connection with the common eigenvectors of SEWs. In each instance, it was shown that the feasible region is a polygon and the Hermitian operators corresponding to half-planes surrounding it are SEWs or positive. Also we have showed that, by using the Clifford group operations one can find vertex points and surrounding half-planes of feasible region only from a few ones.

Appendix I:

Stabilizer theory

Here we summarize the stabilizer formalism and its application to construct an interesting class of EWs so-called *stabilizer entanglement witnesses* (SEWs) [10].

The $l = 2^k$ (where $k = 0, 1, ..., n$) stabilizer states $\{|\psi_1\rangle, ..., |\psi_l\rangle\}$ of n qubits can be thought of as representation of an abelian stabilizer group $\mathcal{S}_{n-k}$ generated by $n-k$ pairwise commuting Hermitian operators in the Pauli group $\mathcal{G}_n$, which consists of tensor products of the identity I_2 and the usual Pauli matrices σ_x, σ_y and σ_z together with an overall phase ± 1 or $\pm i$ [3, 25, 26]. The group $\mathcal{S}_{n-k}$ has 2^{n-k} elements where among them we can choose $S_1, ..., S_{n-k}$ as generators. This group leaves invariant any state in the stabilizer Hilbert space $\mathcal{H}_S$ spanned by $\{|\psi_1\rangle, ..., |\psi_l\rangle\}$, i.e.,

$$S|\psi\rangle = |\psi\rangle \quad ,\forall \ S \in \mathcal{S}_{n-k} \quad , \forall \ |\psi\rangle \in \mathcal{H}_S. \tag{I-1}$$

Similar to the Pauli matrices, for each element S of $\mathcal{S}_{n-k}$ the relation $S^2 = I_{2^n}$ holds and any two elements A_i and A_j of this group satisfy

$$Tr(A_i A_j) = I_{2^n} \delta_{ij} \tag{I-2}$$

The n-qubit Clifford group $Cl(n)$ is the normalizer of $\mathcal{G}_n$ in $U(2^n)$, i.e., it is the group of unitary operators U satisfying $U\mathcal{G}_n U^\dagger = \mathcal{G}_n$. It is a finite subgroup of $U(2^n)$ generated by the Hadamard transform H, the phase-shift gate M, (both applied to any single qubit) and the controlled-not gate $CNOT$ which may be applied to any pair of qubits,

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$
$$CN_{rs}|j\rangle_r|k\rangle_s = |j\rangle_r|j + k \bmod 2\rangle_s \ .$$

Generators of the Clifford group induce the following transformations on the Pauli matrices:

$$\begin{array}{llll} H: & \sigma_x \longrightarrow \sigma_z \quad , & \sigma_y \longrightarrow -\sigma_y \quad , & \sigma_z \longrightarrow \sigma_x \\ M: & \sigma_x \longrightarrow \sigma_y \quad , & \sigma_y \longrightarrow -\sigma_x \quad , & \sigma_z \longrightarrow \sigma_z \end{array} \tag{I-3}$$

$$CN_{12}: \begin{cases} I\otimes\sigma_x \longrightarrow I\otimes\sigma_x, & \sigma_x\otimes I \longrightarrow \sigma_x\otimes\sigma_x, \quad \sigma_y\otimes\sigma_y \longrightarrow -\sigma_x\otimes\sigma_z \\ I\otimes\sigma_y \longrightarrow -\sigma_z\otimes\sigma_y, & \sigma_y\otimes I \longrightarrow \sigma_y\otimes\sigma_x, \quad \sigma_x\otimes\sigma_y \longrightarrow \sigma_y\otimes\sigma_z \\ I\otimes\sigma_z \longrightarrow \sigma_z\otimes\sigma_z, & \sigma_z\otimes I \longrightarrow \sigma_z\otimes I, \quad \sigma_z\otimes\sigma_x \longrightarrow \sigma_z\otimes\sigma_x \end{cases} \tag{I-4}$$

and their actions on the eigenvectors of Pauli operators are

$$\begin{aligned} H|x^{\pm}\rangle &= |z^{\pm}\rangle \\ M|x^{\pm}\rangle &= |y^{\pm}\rangle \\ MH|z^{\pm}\rangle &= |y^{\pm}\rangle, \end{aligned} \tag{I-5}$$

In the following table, we give some examples of stabilizer groups together with the corresponding stabilized states (the states which are invariant under the action of the stabilizer group):

Stabilized state	Generators of stabilizer group
$\vert\psi_{00\ldots0}\rangle$	$S_1^{(\mathrm{GHZ})} := \sigma_x^{(1)}\sigma_x^{(2)}\ldots\sigma_x^{(n)}$ $S_k^{(\mathrm{GHZ})} := \sigma_z^{(k-1)}\sigma_z^{(k)} \qquad k = 2, 3, \ldots, n$
$\vert\mathrm{C}_n\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{a=1}^n \left(\vert 0\rangle_a + \vert 1\rangle_a\ \sigma_z^{(a+1)}\right)$	$S_1^{(\mathrm{C})} = \sigma_x^{(1)}\sigma_z^{(2)}$ $S_k^{(\mathrm{C})} = \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)} \qquad k = 2, 3, \ldots, n-1$ $S_n^{(\mathrm{C})} = \sigma_z^{(n-1)}\sigma_x^{(n)}$
$\vert\Psi_1^{(\mathrm{Fi})}\rangle = \frac{1}{4}\sum_{S\in\mathcal{S}_{Fi}} S\vert 00000\rangle$ $\vert\Psi_2^{(\mathrm{Fi})}\rangle = \frac{1}{4}\sum_{S\in\mathcal{S}_{Fi}} S\vert 11111\rangle$	$S_1^{(\mathrm{Fi})} = \sigma_x^{(1)}\sigma_z^{(2)}\sigma_z^{(3)}\sigma_x^{(4)}$ $S_2^{(\mathrm{Fi})} = \sigma_x^{(2)}\sigma_z^{(3)}\sigma_z^{(4)}\sigma_x^{(5)}$ $S_3^{(\mathrm{Fi})} = \sigma_x^{(1)}\sigma_x^{(3)}\sigma_z^{(4)}\sigma_z^{(5)}$ $S_4^{(\mathrm{Fi})} = \sigma_z^{(1)}\sigma_x^{(2)}\sigma_x^{(4)}\sigma_z^{(5)}$
$\vert\Psi_{ev}^{(\mathrm{Se})}\rangle = \frac{1}{\sqrt{8}}\sum_{\vert\psi\rangle\in\mathcal{E}} \vert\psi\rangle$ $\vert\Psi_{od}^{(\mathrm{Se})}\rangle = \frac{1}{\sqrt{8}}\sum_{\vert\psi\rangle\in\mathcal{O}} \vert\psi\rangle$	$S_1^{(\mathrm{Se})} = \sigma_z^{(1)}\sigma_z^{(3)}\sigma_z^{(5)}\sigma_z^{(7)}$ $S_2^{(\mathrm{Se})} = \sigma_z^{(2)}\sigma_z^{(3)}\sigma_z^{(6)}\sigma_z^{(7)}$ $S_3^{(\mathrm{Se})} = \sigma_z^{(4)}\sigma_z^{(5)}\sigma_z^{(6)}\sigma_z^{(7)}$ $S_4^{(\mathrm{Se})} = \sigma_x^{(1)}\sigma_x^{(3)}\sigma_x^{(5)}\sigma_x^{(7)}$ $S_5^{(\mathrm{Se})} = \sigma_x^{(2)}\sigma_x^{(3)}\sigma_x^{(6)}\sigma_x^{(7)}$ $S_6^{(\mathrm{Se})} = \sigma_x^{(4)}\sigma_x^{(5)}\sigma_x^{(6)}\sigma_x^{(7)}$
$\vert\Psi_{i_1 i_2 i_3}^{(\mathrm{Ei})}\rangle = \left(X_1\right)^{i_1}\left(X_2\right)^{i_2}\left(X_3\right)^{i_3}\sum_{S\in\mathcal{S}_{Ei}} S\vert 0\rangle^{\otimes 8}$ $(i_1, i_2, i_3) \in \{0,1\}^3$	$S_1^{(\mathrm{Ei})} = \sigma_x^{(1)}\sigma_x^{(2)}\ldots\sigma_x^{(8)}$ $S_2^{(\mathrm{Ei})} = \sigma_z^{(1)}\sigma_z^{(2)}\ldots\sigma_z^{(8)}$ $S_3^{(\mathrm{Ei})} = \sigma_z^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}\sigma_z^{(5)}\sigma_x^{(6)}\sigma_y^{(7)}$ $S_4^{(\mathrm{Ei})} = \sigma_z^{(2)}\sigma_z^{(3)}\sigma_x^{(4)}\sigma_x^{(5)}\sigma_y^{(6)}\sigma_y^{(7)}$ $S_5^{(\mathrm{Ei})} = \sigma_x^{(1)}\sigma_x^{(2)}\sigma_z^{(4)}\sigma_y^{(5)}\sigma_y^{(6)}\sigma_z^{(7)}$
$\vert\Psi_{\pm}^{(\mathrm{Ni})}\rangle := \left(\frac{1}{\sqrt{2}}(\vert 000\rangle \pm \vert 111\rangle)\right)^{\otimes 3}$	$S_1^{(\mathrm{Ni})} = \sigma_x^{(1)}\sigma_x^{(2)}\ldots\sigma_x^{(6)} \quad , \quad S_2^{(\mathrm{Ni})} = \sigma_x^{(4)}\sigma_x^{(5)}\ldots\sigma_x^{(9)}$ $S_3^{(\mathrm{Ni})} = \sigma_z^{(1)}\sigma_z^{(2)} \quad , \quad S_4^{(\mathrm{Ni})} = \sigma_z^{(2)}\sigma_z^{(3)}$ $S_5^{(\mathrm{Ni})} = \sigma_z^{(4)}\sigma_z^{(5)} \quad , \quad S_6^{(\mathrm{Ni})} = \sigma_z^{(5)}\sigma_z^{(6)}$ $S_7^{(\mathrm{Ni})} = \sigma_z^{(7)}\sigma_z^{(8)} \quad , \quad S_8^{(\mathrm{Ni})} = \sigma_z^{(8)}\sigma_z^{(9)}$
$\vert\varphi\rangle = \frac{1}{\sqrt{2}}\left(\vert v\rangle + S_1\vert v\rangle\right)$	$S_1^{(\varphi)} = S_1^{(\mathrm{GHZ})} \quad , \quad S_2^{(\varphi)} = \bigotimes_{j=1}^m S_{2j}^{(\mathrm{GHZ})}$

where

$$\mathcal{E} := \{|0000000\rangle, |1010101\rangle, |0110011\rangle, |1101001\rangle, |0001111\rangle, |1100110\rangle, |1011010\rangle, |0111100\rangle\},$$
$$\mathcal{O} := \{|1111111\rangle, |1110000\rangle, |0100101\rangle, |1000011\rangle, |0010110\rangle, |0101010\rangle, |1001100\rangle, |0011001\rangle\},$$
$$X_1 = \sigma_x^{(1)}\sigma_x^{(2)}\sigma_z^{(6)}\sigma_z^{(8)} \quad , \quad X_2 = \sigma_x^{(1)}\sigma_x^{(3)}\sigma_z^{(4)}\sigma_z^{(7)} \quad , \quad X_3 = \sigma_x^{(1)}\sigma_z^{(4)}\sigma_x^{(5)}\sigma_z^{(6)},$$
$$|v\rangle = \left(\sigma_x^{(1)}\right)^{i_1}\left(\sigma_x^{(2)}\right)^{i_2} ... \left(\sigma_x^{(2m)}\right)^{i_{2m}} |0\rangle_1|0\rangle_2...|0\rangle_{2m}, \quad \oplus_{k=1}^{2m} i_k = 0, \quad \forall\ (i_1, i_2, ..., i_{2m}) \in \{0,1\}^{2m}$$

Here $\oplus$ is the sum module 2.

Each of the stabilizer groups stated in the above table, corresponds to some graph states [36, 37]. These states are defined as follows: A graph is a set of n vertices and some edges connecting them. For every graph G, it is associated an adjacency matrix T whose entries are $T_{ij} = 1$ if the vertices i and j are connected and $T_{ij} = 0$ otherwise. Based on that one can attach a stabilizer operator for every vertex i as follows

$$S_i^{(G_n)} = \sigma_x^{(i)} \prod_{j\neq i} \left(\sigma_z^{(j)}\right)^{T_{ij}}$$

The graph state $|G\rangle$ associated with the graph G is the unique n-qubit state satisfying

$$S_i^{(G_n)}|G\rangle = |G\rangle, \qquad \text{for } \ i = 1, ..., n.$$

The case $k = 0$ and $k > 0$ are called graph state and graph code respectively [26, 38]. One can denote the generators of any stabilizer group by a binary $(n-k)\times 2n$ stabilizer matrix $[\mathcal{X}|\mathcal{Z}]$ where $\mathcal{X}$ and $\mathcal{Z}$ are both $(n-k)\times n$ matrices. Matrices $\mathcal{X}$ and $\mathcal{Z}$ have a 1 whenever the generator has a σ_x and σ_z in the appropriate place respectively. For instance, in the five-qubit case, this form becomes

$$[\mathcal{X}|\mathcal{Z}] = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

By adding k rows to the stabilizer matrix $[\mathcal{X}|\mathcal{Z}]$ such that the n resulting rows are linearly independent, one can construct the matrix Γ (called generating matrix in coding theory) as follows

$$\Gamma = \left(\begin{array}{c|c} \mathcal{X} & \mathcal{Z} \\ \hline \tilde{\mathcal{X}} & \tilde{\mathcal{Z}} \end{array}\right)$$

For five-qubit in hand, we have

$$\Gamma_5 = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

It is necessary to note that the added k rows are not unique and this freedom in choice leads to the several locally unitary equivalent graphs for a given graph code.

By using the Gaussian elimination method on matrix Γ one can transform it to the standard form $\Gamma' = [I|A]$, where I is a $n \times n$ identity matrix and $A \equiv T$ is adjacency matrix for the related graph G. The standard form of Γ_5 becomes

$$\Gamma_5 = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

The graphs for the stabilizer groups stated in the above table are shown in Fig. 2.

Appendix II:

Proving the inequalities:

For simplicity in the following proofs we introduce the abbreviations

$$\alpha_i := \langle\sigma_x^{(i)}\rangle \qquad \beta_i := \langle\sigma_y^{(i)}\rangle \qquad \gamma_i := \langle\sigma_z^{(i)}\rangle$$

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 \tag{II-1}$$

where $\langle \sigma_j^{(i)} \rangle$ with $j = x, y, z$ are the expectation values of Pauli operators on any arbitrary pure qubit state.

The proof of (3.11):

We give the proof only for the $j = 2$ since the proof for other cases is similar.

$$|P_1 \pm P_2 + \sum_{k=1}^{n'} (-1)^{i_k} P_{1,2k}| \leq |P_1| + |P_2| + \sum_{k=1}^{n'} |P_{1,2k}| \leq$$

$$|\alpha_1\alpha_2| \left(|\alpha_3...\alpha_n| + \sum_{k=2}^{n'} |\alpha_3\alpha_4...\alpha_{2k-2}\beta_{1,2k-1}\beta_{1,2k}\alpha_{2k+1}...\alpha_n| \right) + |\beta_1\beta_2\alpha_3...\alpha_n| + |\gamma_1\gamma_2| \leq$$

$$|\alpha_1\alpha_2| + |\beta_1\beta_2\alpha_3...\alpha_n| + |\gamma_1\gamma_2| \leq |\alpha_1\alpha_2| + |\beta_1\beta_2| + |\gamma_1\gamma_2| \leq 1$$

The last inequality follows from the Cauchy-Schwartz inequality and (II-1).

To show that the big bracket is smaller than one, we write

$$\left(|\alpha_3...\alpha_n| + \sum_{k=2}^{n'} |\alpha_3\alpha_4...\alpha_{2k-2}\beta_{1,2k-1}\beta_{1,2k}\alpha_{2k+1}...\alpha_n| \right)$$

$$= |\alpha_3...\alpha_n| + |\beta_3\beta_4\alpha_5...\alpha_n| + |\alpha_3\alpha_4\beta_5\beta_6\alpha_7...\alpha_n| + |\alpha_3\alpha_4\alpha_5\alpha_6\beta_7\beta_8\alpha_9...\alpha_n|$$

$$+ ... + |\alpha_3\alpha_4\alpha_5...\alpha_{2n'-2}\beta_{2n'-1}\beta_{2n'}| \leq |\beta_3\beta_4\alpha_5...\alpha_n| + |\alpha_3\alpha_4|$$

$$\times \left[|\alpha_5...\alpha_n| + |\beta_5\beta_6\alpha_7...\alpha_n| + |\alpha_5\alpha_6\beta_7\beta_8\alpha_9...\alpha_n| + ... + |\alpha_5\alpha_6...\alpha_{2n'-2}\beta_{2n'-1}\beta_{2n'}| \right]$$

$$\leq |\beta_3\beta_4\alpha_5...\alpha_n| + |\alpha_3\alpha_4|[|\beta_5\beta_6\alpha_7...\alpha_n|$$

$$+|\alpha_5\alpha_6|[|\alpha_7\alpha_8|[...[|\beta_{2n'-3}\beta_{2n'-2}\alpha_{2n'-1}\alpha_{2n'}| + |\alpha_{2n'-3}\alpha_{2n'-2}|[|\alpha_{2n'-1}\alpha_{2n'}| + |\beta_{2n'-1}\beta_{2n'}]]...]]]$$

By the Cauchy-Schwartz inequality, the last bracket $[|\alpha_{2n'-1}\alpha_{2n'}| + |\beta_{2n'-1}\beta_{2n'}]$ is less than or equal to 1. Replacing it with its maximum value 1, the same argument holds for the term $[|\beta_{2n'-3}\beta_{2n'-2}\alpha_{2n'-1}\alpha_{2n'}| + |\alpha_{2n'-3}\alpha_{2n'-2}|]$. Proceeding in this way, we deduce that the big bracket is less than or equal to 1.

The proof of (3.24):

$$|\gamma_{2m-2}\alpha_{2m-1}\gamma_{2m} \pm \gamma_{2m-3}\alpha_{2m-2}\gamma_{2m-1} + \gamma_{2m-2}\beta_{2m-1}\beta_{2m}\gamma_{2m+1}| \leq$$

$$|\gamma_{2m-2}\alpha_{2m-1}\gamma_{2m}| + |\gamma_{2m-3}\alpha_{2m-2}\gamma_{2m-1}| + |\gamma_{2m-2}\beta_{2m-1}\beta_{2m}\gamma_{2m+1}|$$

taking $\gamma_{2m-3} = \gamma_{2m+1} = 1$

$$\leq |\gamma_{2m-2}\alpha_{2m-1}\gamma_{2m}| + |\alpha_{2m-2}\gamma_{2m-1}| + |\gamma_{2m-2}\beta_{2m-1}\beta_{2m}|$$

Cauchy-Schwartz inequality yields

$$\leq \left(\alpha_{2m-1}^2 + \gamma_{2m-1}^2 + \beta_{2m-1}^2\right)^{\frac{1}{2}} \left(\gamma_{2m-2}^2\gamma_{2m}^2 + \alpha_{2m-2}^2 + \gamma_{2m-2}^2\beta_{2m}^2\right)^{\frac{1}{2}}$$

$$\leq \left(\gamma_{2m-2}^2(\gamma_{2m}^2 + \beta_{2m}^2) + \alpha_{2m-2}^2\right)^{\frac{1}{2}} \leq 1$$

In the above inequalities we have used (II-1) and the fact that

$$a\alpha_i^2 + b\beta_i^2 + c\gamma_i^2 \leq 1 \tag{II-2}$$

when the positive coefficients a, b, c are less than or equal to one. The proof for even case is the same as the odd case.

The proof of (7.57):

We prove only the case $l = 1$.

$$|P_j + P_1 + \sum_{k=1}^{n'} P_{1,2k} + P_{1,3}|$$

$$= |\cos(\theta_{j-1})\cos(\theta_j) + \sin(\theta_1)\sin(\theta_2)...\sin(\theta_n)\{\cos(\phi_1)\cos(\phi_2)...\cos(\phi_n)$$
$$+\sum_{k=1}^{n'}\cos(\phi_1)\cos(\phi_2)...\cos(\phi_{2k-2})\sin(\phi_{2k-1})\sin(\phi_{2k})\cos(\phi_{2k+1})...\cos(\phi_n)$$
$$+\cos(\phi_1)\sin(\phi_2)\sin(\phi_3)\cos(\phi_4)...\cos(\phi_n)\}|$$

$$\leq |\cos(\theta_{j-1})\cos(\theta_j)| + |\sin(\theta_{j-1})\sin(\theta_j)| \times |\cos(\phi_1)\cos(\phi_2)...\cos(\phi_n)$$
$$+\sum_{k=1}^{n'}\cos(\phi_1)\cos(\phi_2)...\cos(\phi_{2k-2})\sin(\phi_{2k-1})\sin(\phi_{2k})\cos(\phi_{2k+1})...\cos(\phi_n)$$
$$+\cos(\phi_1)\sin(\phi_2)\sin(\phi_3)\cos(\phi_4)...\cos(\phi_n)|$$

On the other hand, we note that

$$\cos(\phi_1)\cos(\phi_2)...\cos(\phi_n)+\sum_{k=1}^{n'}\cos(\phi_1)\cos(\phi_2)...\cos(\phi_{2k-2})\sin(\phi_{2k-1})\sin(\phi_{2k})\cos(\phi_{2k+1})...\cos(\phi_n)$$

$$+\cos(\phi_1)\sin(\phi_2)\sin(\phi_3)\cos(\phi_4)...\cos(\phi_n) \leq \frac{1+\sqrt{2}}{2}$$

Hence we get

$$|P_j + P_1 + \sum_{k=1}^{n'} P_{1,2k} + P_{1,3}| \leq |\cos(\theta_{j-1})\cos(\theta_j)| + \frac{1+\sqrt{2}}{2}|\sin(\theta_{j-1})\sin(\theta_j)| \leq \frac{1+\sqrt{2}}{2}$$

The proof of (7.62):

The proof is for $m \geq 2$ and the half-spaces with positive coefficients. The proofs for the other cases are similar.

$$|P_{2m} + P_{2m-1} + P_{2m,2m+1} + P_{2m-1,2m}| =$$

$$|\gamma_{2m-1}\alpha_{2m}\gamma_{2m+1} + \gamma_{2m-2}\alpha_{2m-1}\gamma_{2m} + \gamma_{2m-1}\beta_{2m}\beta_{2m+1}\gamma_{2m+2} + \gamma_{2m-2}\beta_{2m-1}\beta_{2m}\gamma_{2m+1}| \leq$$

$$|\cos(\theta_{2m-1})\sin(\theta_{2m})\cos(\phi_{2m})\cos(\theta_{2m+1}) + \cos(\theta_{2m-2})\sin(\theta_{2m-1})\cos(\phi_{2m-1})\cos(\theta_{2m})$$

$$+\cos(\theta_{2m-2})\sin(\theta_{2m-1})\sin(\phi_{2m-1})\sin(\theta_{2m})\sin(\phi_{2m})\cos(\theta_{2m+1})$$

$$+\cos(\theta_{2m-1})\sin(\theta_{2m})\sin(\phi_{2m})\sin(\theta_{2m+1})\sin(\phi_{2m+1})\cos(\theta_{2m+2})|$$

We note that the maximum value of the right-hand side of the above statement is $\frac{2}{\sqrt{3}}$. Hence we get

$$|P_{2m} + P_{2m-1} + P_{2m,2m+1} + P_{2m-1,2m}| \leq \frac{2}{\sqrt{3}}$$

Appendix III:

Odd case of GHZ SEWs

Let us consider the Hermitian operator

$$\mathcal{W}'^{(n)}_{GHZ} = a_0 I_{2^n} + \sum_{k=1}^{n} a_k S_k^{(\mathrm{GHZ})} + \sum_{k=1}^{n''} a_{1,2k+1} S_1^{(\mathrm{GHZ})} S_{2k+1}^{(\mathrm{GHZ})} \tag{III-1}$$

coming from (3.7) by replacing all even terms with odd ones $S_1^{(\mathrm{GHZ})} S_{2k+1}^{(\mathrm{GHZ})}$ (the name odd refer to the index $2k+1$). It is easily seen that the eigenvalues of $\mathcal{W}'^{(n)}_{GHZ}$ are

$$a_0 + \sum_{j=1}^{n} (-1)^{i_j} a_j + \sum_{k=1}^{n''} (-1)^{i_1 + i_{2k+1}} a_{1,2k+1} \quad , \quad \forall\ (i_1, i_2, ..., i_n) \in \{0,1\}^n \tag{III-2}$$

Product state	$(P_2, P_3, ..., P_{n-1}, P_n, P_1, P_{1,3}, P_{1,5}, ..., P_{1,2n''-1}, P_{1,2n''+1})$
$\|\Psi^{\pm}\rangle$	$(0,0,...,0,0,\pm1,0,0,...,0,0)$
$\Lambda'_1\|\Psi^{\pm}\rangle$	$(0,0,...,0,0,0,\pm1,0,...,0,0)$
$\vdots$	$\vdots$
$\Lambda'_{n''}\|\Psi^{\pm}\rangle$	$(0,0,...,0,0,0,0,0,...,0,\pm1)$
$\Xi_{i_2,...,i_n}\|\Psi^{+}\rangle$	$\big((-1)^{i_2},(-1)^{i_2+i_3},...,(-1)^{i_{n-2}+i_{n-1}},(-1)^{i_{n-1}+i_n},0,0,0,...,0,0\big)$

Table 8: The product vectors and coordinates of vertices for $\mathcal{W}'^{(n)}_{GHZ}$.

The product vectors and the vertex points of the feasible region are listed in the table 8, where

$$\Lambda'_k = \left(M^{(2k)}\right)^{\dagger} M^{(2k+1)}, \qquad k = 1,2,...,n''$$

and $|\Psi^{\pm}\rangle$ is defined as in (3.10). The convex hull of the above vertices, the feasible region, comes from the intersection of the half-spaces

$$|P_1 \pm P_j + \sum_{k=1}^{n''}(-1)^{i_k} P_{1,2k+1}| \leq 1 \quad , \quad j=2,...,n \quad , \quad \forall\ (i_1,i_2,...,i_{n''}) \in \{0,1\}^{n''} \qquad \text{(III-3)}$$

and is a $(n-1)2^{n''+2}$-simplex.

We give the proof only for the $j=2$ since the proof for other cases is similar.

$$|P_1 \pm P_2 + \sum_{k=1}^{n''}(-1)^{i_k} P_{1,2k+1}| \leq |P_1| + |P_2| + \sum_{k=1}^{n''}|P_{1,2k+1}| \leq$$

$$|\alpha_1\alpha_2|\left(|\alpha_3...\alpha_n| + \sum_{k=2}^{n''}|\alpha_3\alpha_4...\alpha_{2k-1}\beta_{1,2k}\beta_{1,2k+1}\alpha_{2k+2}...\alpha_n|\right) + |\beta_1\beta_2\alpha_3...\alpha_n| + |\gamma_1\gamma_2| \leq$$
$$|\alpha_1\alpha_2| + |\beta_1\beta_2\alpha_3...\alpha_n| + |\gamma_1\gamma_2| \leq |\alpha_1\alpha_2| + |\beta_1\beta_2| + |\gamma_1\gamma_2| \leq 1$$

The last inequality follows from the Cauchy-Schwartz inequality and (II-1).

Now the problem is reduced to the following LP problem

$$\begin{array}{ll} \text{minimize} & \mathcal{F}_{\mathcal{W}'^{(n)}_{GHZ}} = a_0 + \sum_{k=1}^{n} a_k P_k + \sum_{k=1}^{n''} a_{1,2k+1} P_{1,2k+1} \\ \text{subject to} & |P_1 \pm P_j + \sum_{k=1}^{n''}(-1)^{i_k} P_{1,2k+1}| \leq 1, \quad j=2,...,n \quad ,\forall\ (i_1,i_2,...,i_{n''}) \in \{0,1\}^{n''} \end{array}$$
$$\text{(III-4)}$$

By putting the coordinates of vertices (see table 2) in the objective function $\mathcal{F}_{\mathcal{W}'^{(n)}_{GHZ}}$ and requiring its non-negativity on all vertices, we get the conditions

$$\begin{array}{c} a_0 > 0 \quad , \quad a_0 \geq |a_1| \quad , \quad a_0 \geq \sum_{i=2}^{n} |a_i| \\ a_0 \geq |a_{1,2k+1}| \qquad k = 1, ..., n'' \end{array} \tag{III-5}$$

on parameters a_i.

Odd case of cluster SEWs

Let us consider the Hermitian operators

$$\mathcal{W}'^{(n)}_{C} = a_0 I_{2^n} + \sum_{k=0}^{n''} a_{2k+1} S^{(\mathrm{C})}_{2k+1} + a_{2m} S^{(\mathrm{C})}_{2m} + a_{2m,2m+1} S^{(\mathrm{C})}_{2m} S^{(\mathrm{C})}_{2m+1} \quad , \quad m = 1, ..., n'' \tag{III-6}$$

Note that instead of the last term we can put the term $a_{2m-1,2m} S^{(\mathrm{C})}_{2m-1} S^{(\mathrm{C})}_{2m}$ with $m = 1, ..., n'$. The eigenvalues of $\mathcal{W}'^{(n)}_{C}$ are

$$a_0 + \sum_{j=0}^{n''} (-1)^{i_{2j+1}} a_{2j+1} + (-1)^{i_{2m}} a_{2m} + (-1)^{i_{2m}+i_{2m+1}} a_{2m,2m+1} \quad , \quad \forall\ (i_1, i_2, ..., i_n) \in \{0,1\}^n \tag{III-7}$$

The product vectors and the vertex points of the feasible region are listed in the table 9

Product state	$(P_1, P_3, ..., P_{2m-3}, P_{2m-1}, P_{2m}, P_{2m+1}, P_{2m+3}, ..., P_{2n''+1}, P_{2m,2m+1})$
$\Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}}\|\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, (-1)^{i_{m-1}}, 0, (-1)^{i_m}, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, 0\big)$
$\Lambda'^{(odd)}_{i_1,i_2,...,i_{n''+1}}\|\Phi\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, \pm 1, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, 0\big)$
$\Lambda''^{(odd)}_{i_1,i_2,...,i_{n''+1}}\|\Phi\rangle$	$((-1)^{i_1}, (-1)^{i_2}, ..., (-1)^{i_{m-2}}, 0, 0, 0, (-1)^{i_{m+1}}, ..., (-1)^{i_{n''+1}}, (-1)^{i_m})$

Table 9: The product vectors and coordinates of vertices for $\mathcal{W}'^{(n)}_{C}$.

where

$$|\Phi\rangle = |z^+\rangle_1 |x^+\rangle_2 |z^+\rangle_3 |x^+\rangle_4 |z^+\rangle_5 ... |x^+\rangle_{n-1} |z^+\rangle_n$$

$$\Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}} = \bigotimes_{j=1}^{n''+1} \left(\sigma_z^{(2j+1)}\right)^{i_j} \bigotimes_{j=1}^{n} H^{(j)} \quad , \quad \forall\ (i_1, i_2, ..., i_{n''+1}) \in \{0,1\}^{n''+1}$$

$$\Lambda'^{(odd)}_{i_1,i_2,...,i_{n''+1}} = \Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}} H^{(2m-1)} H^{(2m)} H^{(2m+1)}$$

$$\Lambda''^{(odd)}_{i_1,i_2,...,i_{n''+1}} = \Lambda^{(odd)}_{i_1,i_2,...,i_{n''+1}} H^{(2m-1)} M^{(2m)} H^{(2m)} M^{(2m+1)}$$

For a given m, the feasible region (the convex hull of the above vertices), comes from the intersection of the half-spaces

$$\begin{aligned}
&|P_{2m} \pm P_{2m-1} + P_{2m,2m+1}| \leq 1 \\
&|P_{2m} \pm P_{2m-1} - P_{2m,2m+1}| \leq 1 \\
&|P_{2m} \pm P_{2m+1} + P_{2m,2m+1}| \leq 1 \\
&|P_{2m} \pm P_{2m+1} - P_{2m,2m+1}| \leq 1 \\
&|P_{2k+1}| \leq 1 \quad , \quad m, m-1 \neq k = 1, ..., n''
\end{aligned} \tag{III-8}$$

and is a $(2n'' + 12)$-simplex. For the first two inequalities we have

$$|\gamma_{2m-1}\alpha_{2m}\gamma_{2m+1} \pm \gamma_{2m-2}\alpha_{2m-1}\gamma_{2m} + \gamma_{2m-1}\beta_{2m}\beta_{2m+1}\gamma_{2m+2}| \leq$$

$$|\gamma_{2m-1}\alpha_{2m}\gamma_{2m+1}| + |\gamma_{2m-2}\alpha_{2m-1}\gamma_{2m}| + |\gamma_{2m-1}\beta_{2m}\beta_{2m+1}\gamma_{2m+2}|$$

taking $\gamma_{2m-2} = \gamma_{2m+2} = 1$

$$\leq |\gamma_{2m-1}\alpha_{2k}\gamma_{2m+1}| + |\alpha_{2m-1}\gamma_{2m}| + |\gamma_{2m-1}\beta_{2m}\beta_{2m+1}| \leq$$

Cauchy-Schwartz inequality yields

$$\leq \left(\alpha_{2m}^2 + \gamma_{2m}^2 + \beta_{2m}^2\right)^{\frac{1}{2}} \left(\gamma_{2m-1}^2\gamma_{2m+1}^2 + \alpha_{2m-1}^2 + \gamma_{2m-1}^2\beta_{2m+1}^2\right)^{\frac{1}{2}}$$

$$\leq \left(\gamma_{2m-1}^2(\gamma_{2m+1}^2 + \beta_{2m+1}^2) + \alpha_{2m-1}^2\right)^{\frac{1}{2}} \leq 1$$

In the above inequalities we have used (II-1) and (II-2).

For the second two inequalities we have

$$|\gamma_{2m-1}\alpha_{2m}\gamma_{2m+1} \pm \gamma_{2m}\alpha_{2m+1}\gamma_{2m+2} + \gamma_{2m-1}\beta_{2m}\beta_{2m+1}\gamma_{2m+2}| \leq$$

$$|\gamma_{2m-1}\alpha_{2m}\gamma_{2m+1}| + |\gamma_{2m}\alpha_{2m+1}\gamma_{2m+2}| + |\gamma_{2m-1}\beta_{2m}\beta_{2m+1}\gamma_{2m+2}|$$

taking $\gamma_{2m-1} = \gamma_{2m+2} = 1$

$$\leq |\alpha_{2m}\gamma_{2m+1}| + |\gamma_{2m}\alpha_{2m+1}| + |\beta_{2m}\beta_{2m+1}| \leq 1$$

The last inequality follows from the Cauchy-Schwartz inequality.

The objective function is

$$\mathcal{F}_{\mathcal{W}'^{(n)}_{C}} = a_0 + \sum_{k=0}^{n''} a_{2k+1}P_{2k+1} + a_{2m}P_{2m} + a_{2m,2m+1}P_{2m,2m+1} \quad , \quad m = 1, ..., n'' \tag{III-9}$$

where

$$P_{2k+1} = Tr(S^{(\mathrm{C})}_{2k+1}\rho_s) \quad , \quad P_{2m,2m+1} = Tr(S^{(\mathrm{C})}_{2m}S^{(\mathrm{C})}_{2m+1}\rho_s),$$

If we put the coordinates of vertices (see table 4) in the objective function (III-9) and require the non-negativity of the objective function on all vertices, we get the conditions

$$\begin{array}{c} a_0 \geq \sum_{j=0}^{n''} |a_{2j+1}| \\ a_0 \geq \sum_{j=1}^{m-2} |a_{2j+1}| + \sum_{j=m+1}^{n''} |a_{2j+1}| + |a_{2m}| \\ a_0 \geq \sum_{j=1}^{m-2} |a_{2j+1}| + \sum_{j=m+1}^{n''} |a_{2j+1}| + |a_{2m,2m+1}| \end{array} \tag{III-10}$$

for parameters a_i.

Exceptional SEWs

Here we mention briefly the SEWs that can be constructed by the stabilizer operations of the five-qubit, seven-qubit, eight-qubit, and nine-qubit states that can be solved by exact LP method.

Five-qubit SEWs

Consider the following Hermitian operator

$$\mathcal{W}_{Fi} = a_0 I_{2^5} + a_1 S_1^{(Fi)} + a_2 S_2^{(Fi)} + a_3 S_3^{(Fi)} + a_{3,4} S_3^{(Fi)} S_4^{(Fi)}$$

Eigenvalues of $\mathcal{W}_{Fi}$ are

$$a_0 + \sum_{j=1}^{3} (-1)^{i_j} a_j \pm a_{3,4} \quad , \quad \forall\ (i_1, ..., i_3) \in \{0, 1\}^3$$

Product state	$(P_1, P_2, P_3, P_{3,4})$
$\lvert\Psi_{\pm}^{(Fi)}\rangle$	$(\pm 1, 0, 0, 0)$
$H^{(2)}H^{(4)}(\mathrm{SW})_{15}\lvert\Psi_{\pm}^{(Fi)}\rangle$	$(0, \pm 1, 0, 0)$
$H^{(3)}H^{(4)}(\mathrm{SW})_{25}\lvert\Psi_{\pm}^{(Fi)}\rangle$	$(0, 0, \pm 1, 0)$
$H^{(1)}H^{(2)}(\mathrm{SW})_{35}\lvert\Psi_{\pm}^{(Fi)}\rangle$	$(0, 0, 0, \pm 1)$

Table 10: The product vectors and coordinates of vertices for $\mathcal{W}_{Fi}$.

The vertex points of the feasible region are listed in Table 10

where

$$|\Psi_{\pm}^{(Fi)}\rangle = |x^{\pm}\rangle_1 |z^{+}\rangle_2 |z^{+}\rangle_3 |x^{+}\rangle_4 | \quad \rangle_5$$
$$(\mathrm{SW})_{ij} = (CN)_{ij}(CN)_{ji}(CN)_{ij}$$

The operator $(SW)_{ij}$ when acts on any two arbitrary pure states swaps them, i.e., $(SW)_{ij}|\psi\rangle_i|\varphi\rangle_j = |\varphi\rangle_i|\psi\rangle_j$. Inequalities obtained from putting the vertex points are

$$a_0 \geq |a_i| \qquad i = 1, 2, 3 \quad , \quad a_0 \geq |a_{3,4}|$$

Boundary half-spaces of feasible region are

$$\begin{aligned} &|P_1 \pm P_2 + P_3 + P_{3,4}| \leq 1 \quad , \quad |P_1 \pm P_2 + P_3 - P_{3,4}| \leq 1 \\ &|P_1 \pm P_2 - P_3 + P_{3,4}| \leq 1 \quad , \quad |P_1 \pm P_2 - P_3 - P_{3,4}| \leq 1 \end{aligned} \tag{III-11}$$

We prove only the following inequality since the proof of the other inequalities is similar to this one.

$$|P_1 + P_2 + P_3 + P_{3,4}| = |\alpha_1\gamma_2\gamma_3\alpha_4 + \alpha_2\gamma_3\gamma_4\alpha_5 + \alpha_1\alpha_3\gamma_4\gamma_5 + \beta_1\alpha_2\alpha_3\beta_4 \leq$$

$$|\alpha_1\gamma_2\gamma_3\alpha_4| + |\alpha_2\gamma_3\gamma_4\alpha_5| + |\alpha_1\alpha_3\gamma_4\gamma_5| + |\beta_1\alpha_2\alpha_3\beta_4| \leq$$

$$|\gamma_3|(|\alpha_1\gamma_2\alpha_4| + |\alpha_2\gamma_4\alpha_5|) + |\alpha_3|(|\alpha_1\gamma_4\gamma_5| + |\beta_1\alpha_2\beta_4|) \leq$$

$$|\gamma_3|\left(\alpha_2^2 + \gamma_2^2\right)^{\frac{1}{2}}\left(\alpha_1^2\alpha_4^2 + \gamma_4^2\alpha_5^2\right)^{\frac{1}{2}} + |\alpha_3|\left(\alpha_1^2 + \beta_1^2\right)^{\frac{1}{2}}\left(\gamma_4^2\gamma_5^2 + \alpha_2^2\beta_4^2\right)^{\frac{1}{2}} \leq$$

$$|\gamma_3|\left(\alpha_1^2\alpha_4^2 + \gamma_4^2\alpha_5^2\right)^{\frac{1}{2}} + |\alpha_3|\left(\gamma_4^2\gamma_5^2 + \alpha_2^2\beta_4^2\right)^{\frac{1}{2}} \leq$$

$$\left(\alpha_3^2+\gamma_3^2\right)^{\frac{1}{2}}\left(\alpha_1^2\alpha_4^2+\gamma_4^2\alpha_5^2+\gamma_4^2\gamma_5^2+\alpha_2^2\beta_4^2\right)^{\frac{1}{2}}\leq$$
$$\left(\alpha_1^2\alpha_4^2+\gamma_4^2(\alpha_5^2+\gamma_5^2)+\alpha_2^2\beta_4^2\right)^{\frac{1}{2}}\leq\left(\alpha_1^2\alpha_4^2+\gamma_4^2+\alpha_2^2\beta_4^2\right)^{\frac{1}{2}}\leq\left(\alpha_4^2+\gamma_4^2+\beta_4^2\right)^{\frac{1}{2}}\leq 1$$

The above inequalities follow from the Cauchy-Schwartz inequality and the equations (II-1) and (II-2).

From 2^4 eigenvalues of $\mathcal{W}_{Fi}$, six of them can take negative values. If we take all $a_1, a_2, a_3, a_{3,4}$ positive and without loss of generality assume that $a_1 \geq a_2 \geq a_3 \geq a_{3,4}$, then these eigenvalues are

$$a_0-a_1-a_2\pm a_3+a_{3,4}\quad,\quad a_0-a_1-a_2\pm a_3-a_{3,4}\quad,\quad a_0\pm a_1\mp a_2-a_3-a_{3,4}$$

Seven-qubit SEWs

Consider the following Hermitian operator

$$\mathcal{W}_{Se}=a_0 I_{2^7}+\sum_{i=1}^{6}a_i S_i^{(Se)}+a_{1,4}S_1^{(Se)}S_4^{(Se)}$$

In addition to the above operator, we can consider other Hermitian operators which differ from the above operator only in the last term, that is the last term of them is one of the following operators

$$S_1^{(Se)}S_4^{(Se)},\ S_2^{(Se)}S_5^{(Se)},\ S_3^{(Se)}S_6^{(Se)},\ S_1^{(Se)}S_5^{(Se)},\ S_2^{(Se)}S_6^{(Se)}$$

Eigenvalues of $\mathcal{W}_{Se}$ are

$$a_0+\sum_{j=1}^{6}(-1)^{i_j}a_j+(-1)^{i_1+i_4}a_{1,4}\qquad\forall\,(i_1,...,i_6)\in\{0,1\}^6$$

The vertex points of feasible region are listed in table 11

where

$$|\Phi^{(Se)}\rangle=|z^+\rangle_1|z^+\rangle_2...|z^+\rangle_7$$
$$\Lambda^{(\mathrm{Se})}_{i_1,i_2,i_3}=\left(\sigma_x^{(1)}\right)^{i_1}\left(\sigma_x^{(2)}\right)^{i_2}\left(\sigma_x^{(4)}\right)^{i_3}$$
$$\Lambda'^{(\mathrm{Se})}_{i_1,i_2,i_3}=\left(\sigma_z^{(1)}\right)^{i_1}\left(\sigma_z^{(2)}\right)^{i_2}\left(\sigma_z^{(4)}\right)^{i_3}\bigotimes_{j=1}^{7}H^{(j)},\quad\forall\,(i_1,i_2,i_3)\in\{0,1\}^3$$
$$\Lambda^{(\mathrm{Se})}_{i}=\left(\sigma_z^{(1)}\right)^{i}\bigotimes_{j=1}^{4}M^{(2j-1)}H^{(2j-1)},\quad\forall\,i\in\{0,1\}$$

Product state	$(P_1, P_2, P_3, P_4, P_5, P_6, P_{1,4})$
$\Lambda^{(\mathrm{Se})}_{i_1,i_2,i_3}\vert\Phi^{(\mathrm{Se})}\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, 0, 0, 0, 0\big)$
$\Lambda'^{(\mathrm{Se})}_{i_1,i_2,i_3}\vert\Phi^{(\mathrm{Se})}\rangle$	$\big(0, 0, 0, (-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, 0\big)$
$\Lambda^{(\mathrm{Se})}_{i}\vert\Phi^{(\mathrm{Se})}\rangle$	$\big(0, 0, 0, 0, 0, 0, (-1)^{i}\big)$

Table 11: The product vectors and coordinates of vertices for $\mathcal{W}_{Se}$.

Boundary half-spaces of feasible region are

$$|P_i \pm P_j + P_{1,4}| \leq 1 \quad , \quad |P_i \pm P_j - P_{1,4}| \leq 1 \qquad i = 1, 2, 3 \quad , \quad j = 4, 5, 6 \tag{III-12}$$

Although all of the inequalities (III-12) can be derived by Cauchy-Schwartz inequality but require a tricky way. The proof of two cases $i = 2, j = 6$ and $i = 3, j = 5$ are similar and therefore we prove only the former case.

$$|P_2 + P_6 + P_{1,4}| = |\gamma_2\gamma_3\gamma_6\gamma_7 + \alpha_4\alpha_5\alpha_6\alpha_7 + \beta_1\beta_3\beta_5\beta_7| \leq$$

$$|\gamma_2\gamma_3\gamma_6\gamma_7| + |\alpha_4\alpha_5\alpha_6\alpha_7| + |\beta_1\beta_3\beta_5\beta_7|$$

taking $\gamma_2 = \alpha_4 = \beta_1 = 1$

$$\leq |\gamma_3\gamma_6\gamma_7| + |\alpha_5\alpha_6\alpha_7| + |\beta_3\beta_5\beta_7| \leq$$

$$\big(\alpha_7^2 + \beta_7^2 + \gamma_7^2\big)^{\frac{1}{2}}\big(\gamma_3^2\gamma_6^2 + \alpha_5^2\alpha_6^2 + \beta_3^2\beta_5^2\big)^{\frac{1}{2}} =$$

$$\big[\gamma_3^2\gamma_6^2(\alpha_5^2 + \beta_5^2 + \gamma_5^2) + \alpha_5^2\alpha_6^2(\alpha_3^2 + \beta_3^2 + \gamma_3^2) + \beta_3^2\beta_5^2(\alpha_6^2 + \beta_6^2 + \gamma_6^2)\big]^{\frac{1}{2}} \leq$$

$$\big[(\alpha_3^2 + \beta_3^2 + \gamma_3^2)(\alpha_5^2 + \beta_5^2 + \gamma_5^2)(\alpha_6^2 + \beta_6^2 + \gamma_6^2)\big]^{\frac{1}{2}} = 1$$

The above inequalities follow from the Cauchy-Schwartz inequality and the equations (II-1) and (II-2).

Inequalities obtained from putting of vertex points are

$$a_0 \geq |a_{1,4}| \quad , \quad a_0 \geq \textstyle\sum_{j=1}^{3} |a_j| \quad , \quad a_0 \geq \textstyle\sum_{j=4}^{6} |a_j|$$

Eight-qubit SEWs

Consider the following Hermitian operator

$$\mathcal{W}_{\mathrm{Ei}} = a_0 I_{2^8} + \sum_{i=1}^{5} a_i S_i^{(\mathrm{Ei})} + a_{1,2,3} S_1^{(\mathrm{Ei})} S_2^{(\mathrm{Ei})} S_3^{(\mathrm{Ei})} + a_{1,2,4} S_1^{(\mathrm{Ei})} S_2^{(\mathrm{Ei})} S_4^{(\mathrm{Ei})}$$

Eigenvalues of $\mathcal{W}_{Ei}$ are

$$a_0 + \sum_{j=1}^{5} (-1)^{i_j} a_j + (-1)^{i_1+i_2+i_3} a_{1,2,3} + (-1)^{i_1+i_2+i_4} a_{1,2,4} \qquad \forall\, (i_1, i_2, ..., i_5) \in \{0,1\}^5$$

The vertex points of feasible region are listed in table 12

Product state	$(P_1, P_2, P_3, P_4, P_5, P_{1,2,3}, P_{1,2,4})$
$\vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(\pm 1, 0, 0, 0, 0, 0, 0\big)$
$H^{(1)} H^{(2)} ... H^{(8)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, \pm 1, 0, 0, 0, 0, 0\big)$
$H^{(1)} M^{(3)} H^{(5)} M^{(7)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, 0, \pm 1, 0, 0, 0, 0\big)$
$H^{(2)} H^{(3)} M^{(6)} M^{(7)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, 0, 0, \pm 1, 0, 0, 0\big)$
$H^{(4)} M^{(5)} M^{(6)} H^{(7)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, 0, 0, 0, \pm 1, 0, 0\big)$
$H^{(2)} M^{(4)} M^{(8)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, 0, 0, 0, 0, \pm 1, 0\big)$
$M^{(1)} H^{(4)} H^{(5)} M^{(8)} \vert\Phi_{\pm}^{(\mathrm{Ei})}\rangle$	$\big(0, 0, 0, 0, 0, 0, \pm 1\big)$

Table 12: The product vectors and coordinates of vertices for $\mathcal{W}_{Ei}$.

where

$$|\Phi_{\pm}^{(\mathrm{Ei})}\rangle = |x^{\pm}\rangle_1 |x^{+}\rangle_2 ... |x^{+}\rangle_8$$

Choosing any seven points among the above vertices give the boundary half-spaces surrounding the feasible region as follows

$$|P_1 + (-1)^{i_1} P_2 + (-1)^{i_2} P_3 + (-1)^{i_3} P_4 + (-1)^{i_4} P_5 + (-1)^{i_5} P_{1,2,3} + (-1)^{i_6} P_{1,2,4}| \le 1 \quad , \quad \forall\, (i_1, i_2, ..., i_6) \in \{0,1\}^6 \tag{III-13}$$

We prove only the following inequality since the proof of the other inequalities of (III-13) is similar to this one.

$$|P_1 + P_2 + P_3 + P_4 + P_5 + P_{1,2,3} + P_{1,2,4}| =$$

$$|\alpha_1\alpha_2...\alpha_8+\gamma_1\gamma_2...\gamma_8+\gamma_1\alpha_2\beta_3\gamma_5\gamma_6\beta_7+\gamma_2\gamma_3\alpha_4\alpha_5\beta_6\beta_7+\alpha_1\alpha_2\gamma_4\beta_5\beta_6\gamma_7+\alpha_1\gamma_2\beta_4\alpha_5\alpha_6\beta_8+\beta_1\alpha_2\alpha_3\gamma_4\gamma_5\beta_8| \leq$$

$$|\alpha_2||\alpha_1\alpha_3...\alpha_8+\gamma_1\beta_3\gamma_5\gamma_6\beta_7+\alpha_1\gamma_4\beta_5\beta_6\gamma_7+\beta_1\alpha_3\gamma_4\gamma_5\beta_8|+|\gamma_2||\gamma_1\gamma_3...\gamma_8+\gamma_3\alpha_4\alpha_5\beta_6\beta_7+\alpha_1\beta_4\alpha_5\alpha_6\beta_8| \leq$$

$$|\alpha_2|\big(|\alpha_1|(\alpha_4^2+\gamma_4^2)^{\frac{1}{2}}(\alpha_3^2\alpha_5^2...\alpha_8^2+{\beta_5}^2{\beta_6}^2\gamma_7^2)^{\frac{1}{2}}+|\gamma_1\beta_3\gamma_5\gamma_6\beta_7|+|\beta_1\alpha_3\gamma_4\gamma_5\beta_8|\big)+$$

$$|\gamma_2|\left(\alpha_4^2+{\beta_4}^2+\gamma_4^2\right)^{\frac{1}{2}}\left(\gamma_1^2\gamma_3^2\gamma_5^2...\gamma_8^2+\gamma_3^2\alpha_5^2{\beta_6}^2{\beta_7}^2+\alpha_1^2\alpha_5^2\alpha_6^2{\beta_8}^2\right)^{\frac{1}{2}} \leq$$

$$|\alpha_2|\left(\alpha_1^2+{\beta_1}^2+\gamma_1^2\right)^{\frac{1}{2}}\left(\alpha_3^2\alpha_5^2...\alpha_8^2+{\beta_5}^2{\beta_6}^2\gamma_7^2+{\beta_3}^2\gamma_5^2\gamma_6^2{\beta_7}^2+\alpha_3^2\gamma_4^2\gamma_5^2{\beta_8}^2\right)^{\frac{1}{2}}+$$

$$|\gamma_2|\left(\gamma_1^2\gamma_3^2\gamma_5^2...\gamma_8^2+\gamma_3^2\alpha_5^2{\beta_6}^2{\beta_7}^2+\alpha_1^2\alpha_5^2\alpha_6^2{\beta_8}^2\right)^{\frac{1}{2}} \leq \left(\alpha_2^2+\gamma_2^2\right)^{\frac{1}{2}} \times$$

$$\left(\alpha_3^2\alpha_5^2...\alpha_8^2+{\beta_5}^2{\beta_6}^2\gamma_7^2+{\beta_3}^2\gamma_5^2\gamma_6^2{\beta_7}^2+\alpha_3^2\gamma_4^2\gamma_5^2{\beta_8}^2+\gamma_1^2\gamma_3^2\gamma_5^2...\gamma_8^2+\gamma_3^2\alpha_5^2{\beta_6}^2{\beta_7}^2+\alpha_1^2\alpha_5^2\alpha_6^2{\beta_8}^2\right)^{\frac{1}{2}} \leq$$

$$\left[\alpha_5^2\left(\alpha_6^2(\alpha_3^2\alpha_7^2\alpha_8^2+\alpha_1^2{\beta_8}^2)+\gamma_3^2{\beta_6}^2{\beta_7}^2\right)+{\beta_5}^2{\beta_6}^2\gamma_7^2+\gamma_5^2\left({\beta_3}^2\gamma_6^2{\beta_7}^2+\alpha_3^2\gamma_4^2{\beta_8}^2+\gamma_1^2\gamma_3^2\gamma_6^2\gamma_7^2\gamma_8^2\right)\right]^{\frac{1}{2}} \leq$$

$$\left(\alpha_5^2+{\beta_5}^2+\gamma_5^2\right)^{\frac{1}{2}} \leq 1$$

where, we have used the Cauchy-Schwartz inequality and the equations (II-1) and (II-2) repeatedly.

Inequalities obtained from putting the vertex points are

$$a_0 \geq |a_i| \qquad i=1,...,5 \quad , \quad a_0 \geq |a_{1,2,3}| \quad , \quad a_0 \geq |a_{1,2,4}|$$

Nine-qubit SEWs

Consider the following Hermitian operator

$$\mathcal{W}_{\mathrm{Ni}} = a_0 I_{2^9} + \sum_{i=1}^{8} a_i S_i^{(\mathrm{Ni})} + a_{1,3} S_1^{(\mathrm{Ni})} S_3^{(\mathrm{Ni})}$$

Eigenvalues of $\mathcal{W}_{Ni}$ are

$$a_0 + \sum_{j=1}^{8}(-1)^{i_j} a_j + (-1)^{i_1+i_3} a_{1,3} \qquad \forall\ (i_1, i_2, ..., i_8) \in \{0,1\}^8$$

Product state	$(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_{1,3})$
$\Lambda^{(\mathrm{Ni})}_{i_1,i_2}\lvert\Phi^{(\mathrm{Ni})}\rangle$	$\big((-1)^{i_1}, (-1)^{i_2}, 0, 0, 0, 0, 0, 0, 0\big)$
$\Lambda^{(\mathrm{Ni})}_{i_1,i_2,i_3}\lvert\Phi^{(\mathrm{Ni})}\rangle$	$\big((-1)^{i_1}, 0, 0, 0, 0, 0, (-1)^{i_2}, (-1)^{i_3}, 0\big)$
$\Lambda'^{(\mathrm{Ni})}_{i_1,i_2,i_3}\lvert\Phi^{(\mathrm{Ni})}\rangle$	$\big(0, (-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, 0, 0, 0, 0, 0\big)$
$\Lambda''^{(\mathrm{Ni})}_{i_1,i_2,i_3}\lvert\Phi^{(\mathrm{Ni})}\rangle$	$\big(0, 0, 0, 0, 0, 0, (-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}\big)$
$\Lambda^{(\mathrm{Ni})}_{i_1,i_2,i_3,i_4,i_5,i_6}\lvert\Phi^{(\mathrm{Ni})}\rangle$	$\big(0, 0, (-1)^{i_1}, (-1)^{i_2}, (-1)^{i_3}, (-1)^{i_4}, (-1)^{i_5}, (-1)^{i_6}, 0\big)$

Table 13: The product vectors and coordinates of vertices for $\mathcal{W}_{Ni}$.

The vertex points of feasible region are listed in table 13

where

$$\lvert\Phi^{(\mathrm{Ni})}\rangle = \lvert x^+\rangle_1 \lvert x^+\rangle_2 ... \lvert x^+\rangle_9$$

$$\Lambda^{(\mathrm{Ni})}_{i_1,i_2} = \left(\sigma_z^{(1)}\right)^{i_1} \left(\sigma_z^{(7)}\right)^{i_2}$$

$$\Lambda^{(\mathrm{Ni})}_{i_1,i_2,i_3} = \left(\sigma_z^{(1)}\right)^{i_1} \left(\sigma_x^{(7)}\right)^{i_2} \left(\sigma_x^{(9)}\right)^{i_3} H^{(7)} H^{(8)} H^{(9)}$$

$$\Lambda'^{(\mathrm{Ni})}_{i_1,i_2,i_3} = \left(\sigma_x^{(1)}\right)^{i_1} \left(\sigma_x^{(3)}\right)^{i_2} \left(\sigma_z^{(4)}\right)^{i_3} H^{(1)} H^{(2)} H^{(3)}$$

$$\Lambda''^{(\mathrm{Ni})}_{i_1,i_2,i_3} = \left(\sigma_z^{(1)}\right)^{i_1} \left(\sigma_x^{(7)}\right)^{i_2} \left(\sigma_x^{(9)}\right)^{i_3} (M^{(1)})^{\dagger} M^{(2)}$$

$$\Lambda^{(\mathrm{Ni})}_{i_1,i_2,i_3,i_4,i_5,i_6} = \left(\sigma_x^{(1)}\right)^{i_1} \left(\sigma_x^{(3)}\right)^{i_2} \left(\sigma_x^{(4)}\right)^{i_3} \left(\sigma_x^{(6)}\right)^{i_4} \left(\sigma_x^{(7)}\right)^{i_5} \left(\sigma_x^{(9)}\right)^{i_6} \bigotimes_{j=1}^{9} H^{(j)}$$

which in all of the above operators we assume that $(i_1, ..., i_j) \in \{0,1\}^j$, with $j = 2,3,6$. By choosing any eight points among the above vertices give the half-spaces surrounding the feasible region as follows

$$\lvert P_1 + P_i \pm P_{1,3}\rvert \le 1 \quad , \quad \lvert P_1 - P_i \pm P_{1,3}\rvert \le 1 \quad , \quad i = 3,4,5,6$$

$$\lvert P_2 \pm P_j\rvert \le 1 \quad , \quad j = 7,8$$

The proof of the above inequalities are straight forward. Inequalities obtained from putting the vertex points are

$$a_0 \ge \lvert a_1\rvert + \lvert a_2\rvert \quad , \quad a_0 \ge \lvert a_1\rvert + \lvert a_7\rvert + \lvert a_8\rvert$$

$$a_0 \ge \lvert a_2\rvert + \lvert a_3\rvert + \lvert a_4\rvert \quad , \quad a_0 \ge \sum_{j=3}^{8} \lvert a_j\rvert \quad , \quad a_0 \ge \lvert a_7\rvert + \lvert a_8\rvert + \lvert a_{1,3}\rvert$$

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Figure Captions

Figure-1: 8-simplex displaying the feasible region of the two-qubit GHZ SEW.

Figure-2: Graphs corresponding to different graph states where the first two ones are graph states and the others are graph codes. (a) The star graph describing a GHZ state. (b) The linear graph describing a cluster state. The graph codes for (c) five-qubit , (d) seven-qubit, (e) eight-qubit and (f) nine-qubit stabilizer groups.